Discrete Math Project 1: Fractions Grow on Trees

due: Friday, March 31, 2023

For this project, you (either by yourself, or with a partner[1]) will carry out a guided mathematical exploration, and write up your results in a nicely typeset document. The purpose of the project is twofold:

[1] Before you ask: no, you may not work in a group of three.

- To solidify your understanding of course topics, by giving you an opportunity to integrate skills and concepts from the course in the context of a larger mathematical exploration.

- To improve your writing skill in mathematics. Mathematics is an inherently social activity; communicating well is an essential part of the practice of mathematics. Clear writing also requires clear thinking: forcing yourself to clearly express your ideas often leads to improved understanding.

Get started early, come ask for help if you need it[2], and have fun!

[2] Or even if you think you might!

LaTeX

You will use LaTeX to typeset your project. LaTeX[3] is used by mathematicians and scientists throughout the world to produce professional-quality technical documents (such as this one). Instead of using a graphical interface to edit exactly what the document looks like (as in Microsoft Word or Apple Pages), a LaTeX document is written as a plain text file with special commands to describe the structure of your document. The document can then be *compiled* to produce a high-quality PDF.

[3] Variously pronounced "LAH-tek", "LAY-tek", "LAH-techhh" (as in Bachhh) or "LAY-techhh". It is NOT pronounced "LAY-tex".

Although it is possible to install LaTeX on your personal computer, there is no need: you can create and edit LaTeX documents for free on `http://overleaf.com`.

- If you do not already have an account on Overleaf, sign up for a free account using your `@hendrix.edu` email address (if you are working with a partner, make sure both of you sign up for an account).

- If you have never used LaTeX before, you should start by spending some time reading/skimming through Overleaf's tutorial, *Learn LaTeX in 30 minutes* (`https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`), and playing around with the LaTeX features it introduces. (Even if you have used LaTeX before, you may learn some new things by skimming through Overleaf's documentation anyway!)

Note: all the URLs in this document are clickable!

- Create a new blank project on Overleaf. Fortunately, a "blank" project is not actually quite blank, and will start you off with the basic skeleton of a document, which you should fill in with your name, document title, date, and so on.

- If you are working with a partner, have one partner create the blank project using their Overleaf account, and then click the "Share" button in the upper right to give the other partner access. Now you can both edit the same document (even at the same time!) through Overleaf.

- Your document will be automatically saved on Overleaf; you can access it from anywhere at any time just by logging in through the Overleaf website.

- If you need help with LaTeX there are many resources available: as a start you can check out the Overleaf documentation (`https://www.overleaf.com/learn`); the Bates LaTeX manual (`https://www.bates.edu/mathematics/resources/latex-manual/`) is also a great resource. You can also ask for help: you can ask me of course, but any math or CS faculty member, as well as many upperclass math or CS majors, should be able to help you as well.

- Check out the LaTeX checklist at `https://github.com/Hendrix-CS/LaTeX-checklist/blob/main/LaTeX-checklist.pdf` to make sure you are using good style!

A Tree of Fractions

Consider building an infinite *tree* of fractions (like a family tree) using the following rules:

- The starting or *root*[4] value is $1/1$.

- Every value a/b has two *children*: the left child of a/b is $a/(a+b)$ and the right child is $(a+b)/b$.

Figure 1 shows the first five levels of the tree (the tree actually continues on infinitely).

- Spend some time at first looking for patterns. What do you notice? What do you wonder? You might want to try extending the tree further, to test out your ideas or just to provide more opportunities for noticing patterns (though you do not necessarily have to include an extended tree in your writeup). Your writeup should include at least one pattern you noticed besides those mentioned in the prompts below.
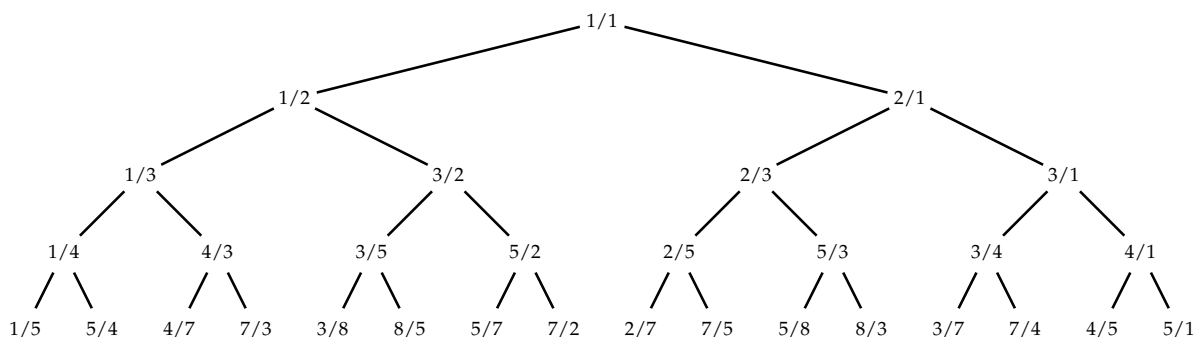
[4] Or perhaps in keeping with the family tree metaphor, we should call it the *progenitor* or *matriarch* or something like that.

Figure 1: The first five levels of the tree

- Prove that every fraction in the tree is in lowest terms, that is, if a fraction a/b shows up somewhere in the tree then a and b share no common factors.[5]

- Prove that every positive rational number occurs in exactly one place in the tree.[6]

- Use the above observations to give an alternate proof that $\mathbb{Q}^+$ (the set of all positive rational numbers) is countable.

- Advanced *optional* challenge: define the *right sibling* of a fraction in the tree to be the next fraction to the right on the same level, or the leftmost fraction on the next lower level for fractions which are already the rightmost fraction on their level. For example, the right sibling of 4/3 is 3/5, and the right sibling of 3/1 is 1/4. Prove either or both of the following:

  - The denominator of each fraction in the tree is the same as the numerator of its right sibling.

  - For any $x \in \mathbb{Q}^+$, the right sibling[7] of x is

$$\frac{1}{1 - x + 2\lfloor x \rfloor},$$

  where $\lfloor x \rfloor$ denotes x rounded down to the nearest integer, that is, the greatest integer less than or equal to x.

[5] Hint: Fhccbfr n/o vf erqhprq gb ybjrfg grezf; jung pna lbh fnl nobhg vgf puvyqera, naq jul?

[6] Hint: Tvira n engvbany ahzore n/o, vs n/o bpphef fbzrjurer va gur gerr, pna lbh svther bhg jung vgf cnerag zhfg or? Vs lbh xrrc qbvat guvf, ubj qb lbh xabj lbh jvyy riraghnyyl ernpu 1/1?

[7] Hint: Qrirybc fbzr sbezhynf sbe zbivat nebhaq va gur gerr. Pna lbh anivtngr guebhtu gur gerr sebz n senpgvba gb vgf evtug fvoyvat?

Assignment

You should create a document with your answers to the above prompts. However, you should *not* just give a bulleted list of answers, and you do not have to answer the questions in exactly the same order as they are listed above. Instead, you should strive to **tell a coherent story** that guides the reader through understanding the tree and the answers to the prompts. Feel free to include any extra information that you feel is helpful to your explanation.

- As the *audience* for your document, you should imagine you are writing to someone who is in Discrete Math but has not done this project or even seen the assignment.

- Since you are writing for someone who has not read the assignment, you should be sure to include the definition of the tree at the beginning of your document! You may copy my definition word-for-word if you wish, or you may explain it in your own words.

- Writing about mathematics does not necessarily have to be dry and technical. You are writing for other humans, not for math robots! Using stories, metaphors, or jokes to make the document more interesting or help the reader understand better is not only allowed, but highly encouraged. If there is some aspect of the tree and its properties that you find exciting or fascinating, you should try to convey your excitement or fascination to the reader.

- You may include pictures or diagrams if you wish, though you are not required to.

- You should use sections to structure your document in some appropriate way.

- Remember to include at least one observation or question about the tree other than those listed in the prompts. You don't necessarily have to prove your observation, but be sure to distinguish between things you *know* are true (with an accompanying proof) and things you *suspect* or *guess* (*i.e. conjecture*) to be true.

Academic Integrity

The document you submit must be your own work (or the work of you and your partner) and give credit where it is due. Specifically:

- You are free to discuss the project with classmates, but your writing must be your own.

  - If you get some good ideas from a classmate, you should cite them! For example, you could add a section called "Acknowledgements" at the end of your document where you say things like "I would like to thank Susan for helping me understand the definition of a rational number, and Faraz for giving me the idea to use complex numbers to prove that every positive rational occurs in the tree.[8]"

  - **I strongly recommend** that you not sit in the same room with others in the class (besides your partner) while typing up your

[8] This is not actually a good idea.

document! It is too tempting to collaborate directly on the structure and phrasing of your writeup. Work together on the ideas and proofs, write some notes, and then later write the document by yourself/yourselves, based on your notes.

- It is possible to find lots of discussion of this tree of fractions on the Internet if you know the right things to search for.[9] I strongly urge you not to look for them (at least not until after turning in your project). You should be able to do the project using only the description of the assigment and things you have learned in class. However, if you do refer to any outside sources, you should be sure to cite them! LaTeX can manage and typeset a bibliography for you; see `https://www.overleaf.com/learn/latex/Bibliography_management_in_LaTeX`. However, you are not required to use a LaTeX bibliography; for this project it will suffice to simply explain in your document what sources you used, including complete URLs.

  [9] I'm not telling.

  Note that you do not need to cite things we have done in class or anything in your textbook.

Grading specification

To receive credit for the project, it must meet the following specifications:

- Mathematics

  - Proofs and explanations are free of logical errors or omissions.
  - Proofs and explanations contain an appropriate level of detail, neither glossing over tricky details nor spending excessive time explaining routine ones.
  - The document answers all the prompts listed in the assignment (other than the optional challenge).
  - Symbols, notation, and terminology are used correctly.

- Writing

  - The document is free of punctuation,, grammatically, and speling erors.
  - The prose flows naturally, using an appropriate variety of vocabulary and sentence structure.
  - The document uses complete sentences; even formulas and equations are included in sentences appropriately.

- Exposition and structure

- – The document does not merely consist of a list of theorems but integrates the results into a coherent story, with each part building on previous parts.
- – The document makes appropriate use of stories, metaphors, *etc.* to engage the reader and help explain relevant concepts.
- – The document is structured in an appropriate way that gives the reader reference points to help them follow along.

• Formatting and typesetting

- – LaTeX formatting is used consistently and appropriately; for example, variables, numbers, and other mathematical expressions are always typeset in "math mode" (using $...$ or \[...\]); special LaTeX syntax is used for quotation marks; there is appropriate space between things.
  See this checklist for a bigger checklist of good LaTeX style:
  `https://github.com/Hendrix-CS/LaTeX-checklist/blob/main/LaTeX-checklist.pdf`

If you are unsure whether your project meets these criteria, you are welcome to bring me a draft (even a partial draft) for some feedback prior to the due date!

Submitting your project

When you are ready to submit your project, you should download the final PDF from Overleaf, and submit it electronically using the assignment submission form linked from the course website. If working with a partner, only one partner needs to submit; make sure both your names are in the document and on the submission form.