MATH 240: Rosen §2.3 #40a sample solution

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40a. Show $f(S \cup T) = f(S) \cup f(T)$.

To prove this we first have to make sure we understand all the concepts involved!

1. First, what does $f(S)$ mean? A function f being applied to a whole set? In general, if $f : A \to B$ and $S \subseteq A$ then

$$f(S) = \{f(s) \mid s \in S\},$$

that is, $f(S)$ is the set of all the outputs we get if we apply f to everything in S. $f(S)$ is called the *image* of S under f.

2. Next, how do we show two sets are equal? To show $A = B$ we must show that $A \subseteq B$ and $B \subseteq A$.

3. Well, how do we show $A \subseteq B$? By definition, $A \subseteq B$ means $\forall a.\ a \in A \to a \in B$. So we must show for any arbitrary $a \in A$ that a is also an element of B.

4. Finally, what is the definition of $S \cup T$?

$$S \cup T = \{x \mid x \in S \vee x \in T\}.$$

We can now put all these together to do part of the proof. Let $f : A \to B$ and $S, T \subseteq A$. Let's show that $f(S \cup T) \subseteq f(S) \cup f(T)$ (this is one half of the proof).

Proof. Let $x \in f(S \cup T)$; then we must show that $x \in f(S) \cup f(T)$. By definition of a function applied to a set, if $x \in f(S \cup T)$, then there exists some $s \in S \cup T$ such that $f(s) = x$. Then by definition of $\cup$, either $s \in S$ or $s \in T$. We consider both cases.

- If $s \in S$, then $x \in f(S)$ since $x = f(s)$. Therefore $x \in f(S) \cup f(T)$.
- The case for $s \in T$ is similar.

□