MATH 240 Module 9: Combinatorics

due Friday, 28 April 2023

Learning Goals

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Submission

You should submit:

- a PDF with your answers to the exercises (you may either type your answers and export as a PDF, or write your answers by hand and scan them using an app such as GeniusScan or CamScanner).

You are welcome to use Disco to help you on this module, but you are not required to turn in any `.disco` files.

Exercises

Exercise 1 How many bit strings are there of length 6 or less, not counting the empty string?

Exercise 2 How many strings of eight uppercase English letters are there ...

(a) if letters can be repeated?

(b) if no letter can be repeated?

(c) that start with X, if letters can be repeated?

(d) that start with X, if no letter can be repeated?

(e) that start and end with X, if letters can be repeated?

(f) that start with the letters BO (in that order), if letters can be repeated?

(g) that start and end with the letters BO (in that order), if letters can be repeated?

(h) that start or end with the letters BO (in that order), if letters can be repeated?

Exercise 3 How many ways are there for 10 dogs and 6 cats to stand in a line so no two cats stand next to each other?

Hint: first position the dogs and then consider possible positions for the cats.

Exercise 4 You are a mythical creature trainer! You have seven dragons and nine unicorns, and it's time to select a team to compete in the Annual Creatures of Myth Exposition (ACME).

(a) How many ways are there to choose a team of five creatures if at least one dragon must be on the team?

Hint: How many possible teams are there? How many teams are there with no dragons?

(b) How many ways are there to choose a team of five creatures if at least one dragon and at least one unicorn must be on the team?

Exercise 5 (Level 2) (**Optional challenge, +1/2 token**) Suppose p and q are primes, and let $n = pq$.

(a) How many positive integers less than or equal to n are divisible by p? How many are divisible by q?

(b) Use the subtraction rule (*i.e.* the Principle of Inclusion-Exclusion) to find the number of positive integers less than or equal to n which share a common factor with n.

(c) How many positive integers less than or equal to n are relatively prime to (*i.e.* do *not* share any common factors with, *i.e.* have a gcd of 1 with) n?

(d) The *Euler totient function* $\varphi(n)$ denotes the number of positive integers $\leq n$ which are relatively prime to n. For example, $\varphi(12) = 4$ since 1, 5, 7 and 11 are the only positive integers ≤ 12 which share no common factors with 12. Use your answer to the previous part to show that $\varphi(pq) = \varphi(p) \cdot \varphi(q)$.

Exercise 6 (**Optional challenge, +1/2 token**) Prove that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

using a *combinatorial* argument instead of algebra—that is, explain why the two sides of the equation represent two different ways to count the same set of objects.