MATH 240 Module 8: Sequences and Induction

due Monday, 17 April 2023

Learning Goals

- Describe integer sequences in terms of both recurrences and closed forms.
- Use the principle of induction to prove propositions about natural numbers.

Submission

You should submit:

- a PDF with your answers to the exercises (you may either type your answers and export as a PDF, or write your answers by hand and scan them using an app such as GeniusScan or CamScanner).
- `module8.disco`. Some of the exercises on this module require you to write Disco code, but unlike previous modules, I have not given you a starting `.disco` file to fill in. You should create your own `.disco` file (feel free to use `.disco` files from previous modules as examples). You are not required to write any documentation or tests for your functions, although you are encouraged to do so since you may find it helpful.

Exercises

Exercise 1 For each recurrence below, list at least the first 5 elements of the sequence, and come up with a closed form. You do not have to prove that your closed form is correct, though you may if you want some extra practice with induction.

(a) $a_n = a_{n-1} + 2; a_0 = 3$

(b) $a_n = 5a_{n-1}; a_0 = 1$

(c) $a_n = a_{n-1} + (2n-1); a_0 = 0$

(d) $a_n = 2na_{n-1}; a_0 = 3$

Exercise 2 This exercise concerns the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}.$$

(a) Write a Disco function `fracsum :  N -> F` which computes the above sum for a given n. For example, `fracsum(2)` should output the sum $1/(1\cdot 2)+1/(2\cdot 3)$.

Hint: your function should be recursive. Make sure your function is defined for *every* natural number input, including 0.

(b) Evaluate your function for some example inputs and look for a pattern. Make a conjecture of the form

$$\forall n : \mathbb{N}. \mathit{fracsum}(n) = \ldots.$$

(c) Use induction to prove your conjecture.

Exercise 3 Prove that any natural number can be written as a sum of distinct powers of two, that is,

$$n = 2^a + 2^b + 2^c + \ldots$$

where the exponents $a, b, c, \ldots$ are all different.

Hint: use a proof by strong induction. In the induction step, consider separately the cases where $n+1$ is even or odd. If $n+1$ is even, note that $(n+1)/2$ must be an integer.

Exercise 4 This exercise concerns a variant of the *Ackermann function* (originally due to R. C. Buck), defined on pairs of natural number inputs as follows:

$$A(m,n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

(a) Find $A(1,0)$, $A(0,1)$, $A(1,1)$, and $A(2,2)$.

(b) Prove that $A(m,2) = 4$ for all $m \geq 1$.

(c) Prove that $A(1,n) = 2^n$ for all $n \geq 1$.

(d) Find each of the following values.

You might be able to use Disco to find some of them; for others, you can use the above theorems to help you calculate a result.

(i) $A(2,3)$

(ii) $A(3,3)$

(iii) $A(3,4)$ (*Hint:* feel free to leave your answer in a convenient algebraic form rather than expanding it out into its actual decimal digits.)