MATH 240 Module 5: Functions

due Friday, 10 March 2023

Learning Goals

- Define and use functions in Disco.
- Describe functions that are or are not 1-1 or onto.
- Prove that a given function is or is not 1-1 or onto.

Submission

You should submit two files:

- A PDF with your answers to the exercises (you may either type your answers and export as a PDF, or write your answers by hand and scan them using an app such as GeniusScan or CamScanner). Note that you **must** submit a PDF! Submissions in any other format (`.docx`, `.pages`, ...) will need to be resubmitted. If you are not sure how to create a PDF document please ask for help!
- A completed version of `module5.disco`; see `https://replit.com/@BrentYorgey/Discrete-Math-Module-5`.

Exercises

Exercise 1 For each of the following functions, determine whether the function is injective, surjective, both, or neither, and give an informal justification for your determination. Feel free to use Disco to help explore the behavior of these functions, though you are not required to do so.

(a) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x - 1$

(b) $f : \mathbb{N} \to \mathbb{Z}; f(x) = x - 1$

(c) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x^3$

(d) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; f(a, b) = a + b$

Exercise 2 Suppose that $f : A \to B$ and $g : B \to A$ are functions. This question explores the idea of showing that f is a bijection by demonstrating that it has an inverse.

(a) Prove that if $\forall a : A.\ g(f(a)) = a$, then f is injective (1-1).

(b) Prove that if $\forall b : B.\ f(g(b)) = b$, then f is surjective (onto).

(c) (**Optional challenge, +1/2 token**): Prove the converse, that is, if $f : A \to B$ is 1-1 and onto, then it has an inverse.

Hint: use the fact that f is onto to define an appropriate function g; then show that $g(f(a)) = a$ and $f(g(b)) = b$.

Exercise 3 Let A, B, and C be arbitrary sets. If $f : A \to B$ and $g : B \to C$ are functions, the *composition* $g \circ f : A \to C$ is the function defined by

$$(g \circ f)(a) = g(f(a)).$$

Prove that if f and g are bijections, then so is $g \circ f$.

Exercise 4 (**Optional challenge, +1/2 token**) Let $f : A \to B$ and $g : C \to D$ be bijections. Give an appropriate definition of a function $(f \times g) : A \times C \to B \times D$ and prove that it is also a bijection.

Disco

You must also complete Disco programming **Exercises D1 and D2**: see `https://replit.com/@BrentYorgey/Discrete-Math-Module-5`.