MATH 240 Module 2: Propositional logic and quantifiers

due Friday, 10 Feb 2023

Learning Goals

- Interpret propositions with quantifiers
- Translate English sentences into formal propositional logic notation using nested quantifiers
- Express quantified statements in equivalent ways with different domains
- Use De Morgan laws to simplify negations of propositions with quantifiers, AND, and OR

Submission

You should submit:

- A PDF with your answers to the exercises (you may either type your answers and export as a PDF, or write your answers by hand and scan them using an app such as GeniusScan or CamScanner). Note that you **must** submit a PDF! Submissions in any other format (`.docx`, `.pages`, ...) will need to be resubmitted. If you are not sure how to create a PDF document please ask for help!

Exercises

Exercise 1 Express the negation of each proposition below so that all negation symbols immediately precede predicate variables. In other words, "push negation inwards as far as possible". For example, given the proposition $\forall x : D.\ P(x) \vee Q(x)$, we could express its negation as

$$\exists x : D. \neg P(x) \wedge \neg Q(x).$$

(a) $\forall x{:}D.\ T(x) \vee G(x) \vee F(x)$

(b) $\forall x{:}D.\ \exists y{:}E.\ P(x) \wedge Q(x,y) \wedge R(y)$

(c) $(\exists x{:}\mathbb{Z}.\ P(x)) \vee (\forall y{:}\mathbb{Z}.\ Q(x))$

(d) $\forall x{:}S.\ \exists y{:}D.\ H(x,y) \rightarrow (P(x) \vee Q(y))$

(e) $\forall m{:}D.\ \forall n{:}D.\ P(m,n) \leftrightarrow Q(m,n)$

Exercise 2 For each English sentence below:

- Translate it into formal propositional logic using quantifiers.
- Decide if it is true or false. Justify your answer using a truth table, algebraic reasoning, and/or an informal logical argument.
- Use Disco to check your answer with `:test`. In which cases do you trust Disco's answer? Were there any cases where Disco helped you change your mind?

(a) There is an integer n such that 20 is 7 more than n.

(b) For every integer n, 7 more than n is less than 20.

(c) There are two integers which add to 6.

(d) For any propositions p, q, and r, if p implies q and q implies r, then p implies r.

(e) For every proposition p, there is a proposition q such that $(p \wedge q) \rightarrow \text{True}$.

(f) For every proposition p, there is a proposition q such that $p \wedge q$ is logically equivalent to True.

Exercise 3 Translate each English sentence below into formal propositional logic, using quantifiers as appropriate. You are welcome to use the predicates $\mathrm{Sq}(n)$ = "n is a perfect square" and $\mathrm{Even}(n)$ = "n is even" which we defined in class.

(a) If n is a perfect square, $n + 2$ is not a perfect square.

(b) If mn is even, then either m is even or n is even (or both).

(c) Any integer n is even if and only if $7n + 4$ is even.

(d) Every natural number can be written as the sum of two squares of natural numbers.

Exercise 4 Say whether each proposition below is true or false. Justify your answers.

(a) $\exists a : \mathbb{Z}.\ \forall b : \mathbb{Z}.\ a \cdot b = a$

(b) $\forall x : \mathbb{N}.\ \forall y : \mathbb{N}.\ \forall z : \mathbb{N}.\ (x + y) + z = x + (y + z)$

(c) $\forall a : \mathbb{Z}.\ \exists q : \mathbb{N}.\ q^2 = a$

(d) $\forall a : \mathbb{Z}.\ \exists b : \mathbb{Z}.\ \forall c : \mathbb{Z}.\ a + b + c = c$

Exercise 5 Explain how we can rewrite the following proposition into a logically equivalent proposition using only quantifiers over $\mathbb{Z}$, that is, using $\forall$ and $\exists$ with domain $\mathbb{Z}$ instead of the more restricted domain Even.

$$\forall x : \mathrm{Even}.\ \exists y : \mathrm{Even}.\ x = y + 2$$