The addition rule

Suppose choosing something means <u>either</u> making one choice in c_1 ways <u>or</u> making a different choice in c_2 ways, <u>and</u>, the choices do not overlap at all. Then the total number of choices is

$$c_1 + c_2 .$$

In other words, when A and B are disjoint (no overlap)

$$|A \cup B| = |A| + |B|$$

<u>Ex.</u> Want lunch —

$\left.\begin{array}{l}\text{Restaurant A has 13 menu items} \\ \text{Restaurant B has 24 menu items}\end{array}\right\}$ no overlap.

Total # of lunch choices is $13 + 24 = 37$.

<u>Ex.</u> 10 soups, want to make a schedule for <u>up to</u> 4 days. (1 to 4).

We can count # schedules for each # of days separately, then add.

$$\underbrace{10}_{\text{\# of 1-day schedules}} + \underbrace{10 \cdot 9}_{\text{\# of 2-day schedules}} + 10 \cdot 9 \cdot 8 + 10 \cdot 9 \cdot 8 \cdot 7$$

etc.

Alternatively:

- 10 choices for the first day
- AND choose what to do for the remaining 3 days:
  - Not eat any more soup (1 choice)

  OR

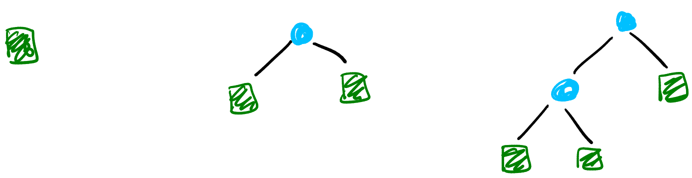
  - Eat a soup (9 choices) AND choose what to do for the remaining 2 days.

$$10\cdot(1+9\cdot(1+8\cdot(1+7)))$$

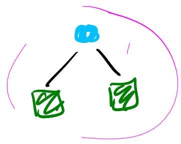
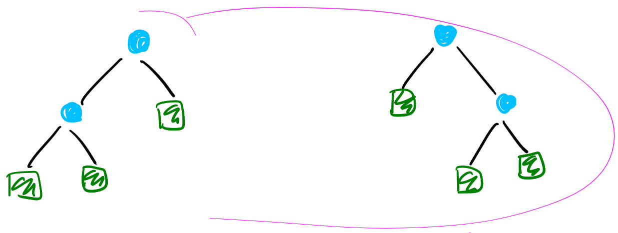
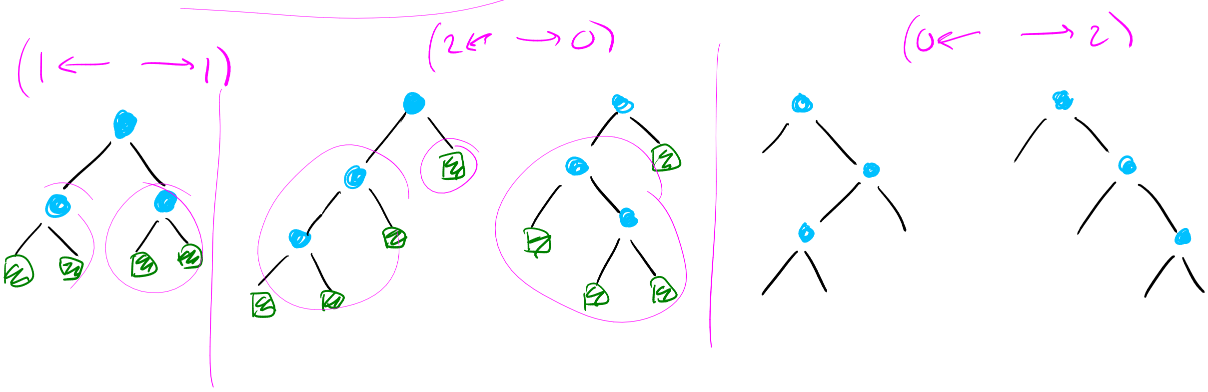
Ex.

Defn: A binary tree is either

- A "Leaf", or
- A "branch" with left + right subtrees which are both binary trees.



Q: How many binary trees are there with n branch nodes?

n	Trees w/ n branch nodes
0	
1	
2	
3	

In general, to make a tree, we

1. Put 1 branch node @ the root
2. Decide how many branch nodes go on L vs. R ← addition rule
3. Choose 2 subtrees of appropriate sizes. ← product rule

Let $T(n)$ be # of binary trees w/ n branch nodes.

$$T(0) = 1$$
$$T(1) = 1$$
$$T(2) = 2 \qquad \ldots$$

$$T(3) = \underbrace{T(2)\cdot T(0)}_{\text{tree w/ 2 on left AND 0 on right}} \overset{\text{OR}}{+} \underbrace{T(1)\cdot T(1)}_{\text{1 on left AND 1 on right}} + T(0)\cdot T(2)$$
$$= 2\cdot 1 + 1\cdot 1 + 1\cdot 2 = 5$$

$$T(4) = T(3)\cdot T(0) + T(2)\cdot T(1) + T(1)\cdot T(2) + T(0)\cdot T(3)$$
$$= 5\cdot 1 + 2\cdot 1 + 1\cdot 2 + 1\cdot 5$$
$$= 14.$$

$$T(5) = T(4)\cdot T(0) + T(3)\cdot T(1) + T(2)\cdot T(2)$$
$$+ T(1)\cdot T(3) + T(0)\cdot T(4)$$
$$= 14\cdot 1 + 5\cdot 1 + 2\cdot 2 + 1\cdot 5 + 1\cdot 14$$
$$= 42.$$

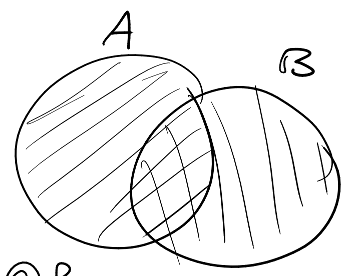
1, 1, 2, 5, 14, 42, 132, 429, ...

<u>Catalan numbers</u>

The subtraction rule

If choosing something means making one choice in c_1 ways, OR another choice in c_2 ways, the total # of choices is

$$c_1 + c_2 - \underline{(\text{\# of choices in common})}$$



ie. $$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex. Lunches, again: 13 choices @ A, 24 choices @ B, but 9 on both. Total choices $= 13 + 24 - 9$.

Ex. How many strings of 8 bits either start w/ 1 or end w/ 00?

- # strings start w/ 1? 2^7 — 1 choice for 1st bit, 2 for each subsequent bit.
- # strings end w/ 00? 2^6.
- # strings start w/ 1 AND end w/ 00? 2^5.

Hence, total is $\underline{2^7 + 2^6 - 2^5}$. $(= 160)$

(generalizes to Principle of Inclusion-Exclusion, eg.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
)