Combinatorics

Combinatorics = counting things, aka determining cardinality of finite sets.

Product rule

Suppose choosing an item can be decomposed into two independent choices, with c_1 ways to make the first choice, and c_2 ways to make the second. Then the total # of ways to choose overall is $c_1 \times c_2$.

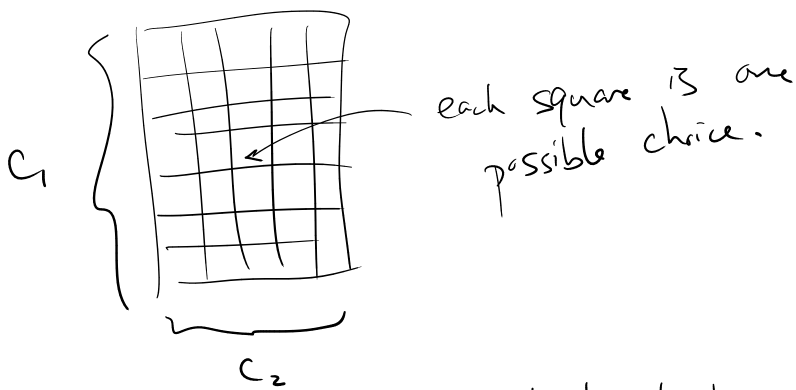
We will often talk in terms of <u># of ways to choose</u> something, rather than how many there are. But it is equivalent.

<u>Independent</u> = the two choices do not affect each other at all — same set of choices no matter what is chosen for the other choice.

<u>Ex.</u> There are three kinds of cereal and 7 kinds of fruit. For breakfast, you will choose one of each. How many different breakfasts could you have? Answer: $3 \times 7 = 21$.

<u>Ex.</u> Suppose same scenario, but you hate bananas on oatmeal. Product rule does not directly apply, since choices are no longer independent.

In general, we can visualize choices in a 2D grid:



This is really just saying $|A \times B| = |A| \times |B|$.

Ex. How many 3-letter strings are there using the letters A-Z?

e.g. AAA, CAR, HDX, ...

independent because repeated letters allowed.

- 26×26 ways to choose 2 letters
- $(26^2) \times 26$ ways to choose (2 letters) + 1 more letter.

↳ generalized product rule:

$c_1 \times c_2 \times c_3 \times \dots \times c_n$ ways to make n independent choices.

- in general, 26^n ways to choose string of length n.

Ex. How many subsets are there of an n-element set?

Each element can be independently chosen to be either in or out of a subset. Hence, by product rule, there are

$$\underbrace{2 \times 2 \times 2 \times \dots \times 2}_{n} = 2^n$$

ways to choose a subset.

Ex. Suppose we have 10 distinct cans of soup and we will eat them all, one per day. How many ways can we do this? (i.e. how many schedules?)

Product rule does not apply directly, because which soup we choose one day affects our choices on subsequent days.

But we can think in terms of independent choices as long as we account for the fact that we have 1 fewer choice each day.

Hence total choices $= 10 \times 9 \times 8 \times \dots \times 1 = 10!$

Note this = same as asking how many different orders (permutations) we could put cans in.

More generally, there are $n!$ permutations of n things.

Ex Same situation w/ 10 soups, but we will only eat soup for 4 days.

$$10 \cdot 9 \cdot 8 \cdot 7$$

$$\left(= \frac{10!}{6!} \right)$$

In general, if we have n things and want to pick k of them in a particular order, # of choices is

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

Ex. Given finite sets A and B, how many different functions are there $f : A \to B$?

For each element $a \in A$, we can independently choose which element of b should be the corresponding output.

Hence the total is

$$\underbrace{|B| \times |B| \times \cdots \times |B|}_{|A|} = |B|^{|A|}.$$

eg. there are 2^{26} functions from $\{A, \ldots, Z\} \to \text{Bool}$,

or 3^2 functions $\text{Bool} \to \{0, 1, 2\}$