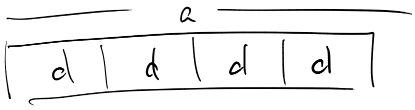
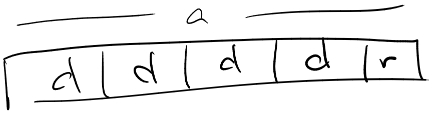
Recall: $d \mid a$ means $k \cdot d = a$ for some $k$.




$d \nmid a$




**Theorem** (The Division Algorithm)

For all $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, there exist unique integers $q$ and $r$ such that:

positive integer, ie $\{1,2,3,\ldots\}$

$$a = dq + r \quad \leftarrow \quad a = q \text{ copies of } d + \text{leftover}$$

$$0 \le r < d. \quad \leftarrow \quad \text{leftover bit is } < d.$$

$q$ is called the quotient, $r$ is called the remainder, and we write

$q = a \text{ div } d$ (Java, C/C++ use /, Python, Disco use //)

$r = a \text{ mod } d.$ (Java, C/C++, Python ... use %, Disco uses mod)

**Examples**

① What are the quotient + remainder when 101 is divided by 11?

$101 \text{ div } 11 = 9$

$101 \text{ mod } 11 = 2$

check: $101 = 9 \cdot 11 + 2$ ✓

$\boxed{0 \le 2 < 11}$

② 55 divided by 11?

$55 \text{ div } 11 = 5$

$55 \text{ mod } 11 = 0.$

③

$7 \text{ mod } 11 = 7$

$7 = 0 \cdot 11 + 7$

$7 = 1 \cdot 11 - 4$

but $-4$ not in range $0 \le r < 11.$

(4) Quotient + remainder when dividing $-24$ by $11$?

$-24 \text{ div } 11 = -3$ $\qquad$ $-3 \cdot 11 + 9 = -24$

$-24 \text{ mod } 11 = 9$ $\qquad$ $0 \leq 9 < 11$

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Why is this true?

Basic idea: Given $a$, $d$, start with

$q = 0$
$r = a$ $\Big\}$ $q \cdot d + r = 0 \cdot d + a = a$

BUT $r$ probably not in required range.

As long as $r$ is too big,

- add 1 to $q$
- subtract $d$ from $r$

Likewise, if $r$ too small, subtract 1 from $q$ and add $d$ to $r$.

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Theorem Every integer is either even or odd. That is,

$$\forall n : \mathbb{Z}.\ \mathrm{Even}(n) \vee \mathrm{Odd}(n).$$

Proof Let $n$ be arbitrary integer. By Division Algorithm, $n = 2q + r$ where $0 \leq r < 2$. Either $r = 0$, in which case $n$ is even by definition, or $r = 1$, in which case $n$ is odd by definition.