Induction.

Q: How do we prove a $\forall$?

1. Prove for arbitrary element of domain.
2. Induction!

We will focus on induction for $\mathbb{N}$ as a domain (it is actually more general).

Idea: have proposition $P(n)$. Want to prove $\forall n : \mathbb{N}. P(n)$.

$$P(0) \longrightarrow P(1) \longrightarrow P(2) \longrightarrow P(3) \longrightarrow P(4) \longrightarrow \cdots$$

1. ("Base case"): prove $P(0)$ is true.
2. ("Induction step"): prove that $P(k) \to P(k+1)$ for all k.
   (by supposing k is arbitrary, suppose $P(k)$, prove $P(k+1)$)

Then $P(n)$ is true for all $n : \mathbb{N}$.

Formally, the principle of (weak) (natural number) induction states:

$$\Big[\underbrace{P(0)}_{\text{base case}} \wedge \overbrace{\big(\forall k : \mathbb{N}.\ \underbrace{P(k)}_{\text{induction hypothesis}} \to P(k+1)\big)}^{\text{induction step}} \Big] \longrightarrow \big(\forall n : \mathbb{N}.\ P(n)\big).$$

Ex. Prove $1+2+3+\dots+n = \frac{n(n+1)}{2}$ for all $n \geq 0$.

We have seen 2 proofs of this already; let's see a proof by induction.

Proof. Let $P(n)$ be the proposition

$$\sum_{1\leq j\leq n} j = \frac{n(n+1)}{2}.$$

We want to show $\forall n:\mathbb{N}.\ P(n)$, so we can use induction.
We must therefore prove $P(0) \wedge (\forall k:\mathbb{N}.\ P(k) \to P(k+1))$. We will prove each.

$P(0)$ is the proposition

$$\sum_{1\leq j\leq 0} j = \frac{0(0+1)}{2} = 0,$$ which is true.

also 0, no j's to add up.

To show $\forall k:\mathbb{N}.\ P(k) \to P(k+1)$, let k be an arbitrary natural number, and suppose $P(k)$, that is,

$$\sum_{1\leq j\leq k} j = \frac{k(k+1)}{2}.$$ (KNOW) (induction hypothesis)

Then we must show $P(k+1)$, that is,

$$\sum_{1\leq j\leq k+1} j = \frac{(k+1)(k+2)}{2}.$$ (WANT)

$$\sum_{1\leq j\leq k+1} j$$

$=$ {take out $j=k+1$ term}

$$(k+1) + \sum_{1\leq j\leq k} j$$

$$= \quad \{\text{induction hypothesis}\}$$

$$(k+1) + \frac{k(k+1)}{2} \quad \overset{=}{\longrightarrow} \quad \frac{2(k+1)}{2} + \frac{k(k+1)}{2}$$

$$\{\text{algebra}\}$$

$$= \frac{2(k+1) + k(k+1)}{2}$$

$$\{\text{factor } k+1\}$$

$$= \frac{(2+k)(k+1)}{2}$$

$$\{\text{algebra}\}.$$

$$= \frac{(k+1)(k+2)}{2}$$

Since we showed both $P(0)$ and that $P(k) \to P(k+1)$ for all k, by the principle of induction, $P(n)$ is true for all natural #'s.

$\boxtimes$.

Ex. $P(n) \equiv (2^n < n!)$

$P(0) \equiv (2^0 < 0!)$ ✗ $1 \not< 1$

$P(1) \equiv (2^1 < 1!)$ ✗ $2 < 1$

$P(2) \equiv (2^2 < 2!)$ ✗ $4 < 2$

$P(3) \equiv (2^3 < 3!)$ ✗ $8 < 6$.

$P(4) \equiv (2^4 < 4!)$ ✓ $16 < 24$

$P(5) \equiv (2^5 < 5!)$ ✓ $32 < 120$

Conjecture: $P(n)$ is true for all $n \geq 4$.

We can use a generalized form of induction that lets us start at a different base case. (Or prove $\forall n : \mathbb{N}.\ Q(n)$, where $Q(n) \equiv P(n+4)$.)

Proof.

Base case: P(4) says $2^4 < 4!$, which is true: $16 < 24$.

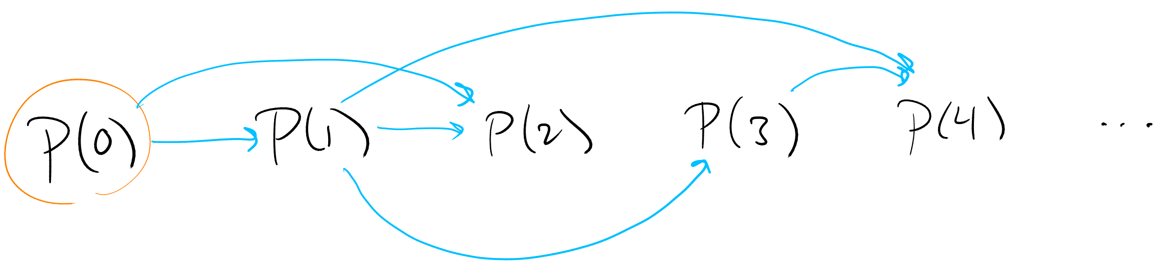
Induction step: Let k be an arbitrary natural number ≥ 4, and suppose $P(k)$, that is, $\boxed{2^k < k!}$. know Then we must show $P(k+1)$, that is, $\boxed{2^{k+1} < (k+1)!}$. want.

$$
\begin{aligned}
& 2^{k+1} \\
= \quad & \{\text{algebra}\} \\
& 2 \cdot 2^k \\
< \quad & \{\text{induction hypothesis, } 2^k < k!, \text{ and mult. is monotonic}\}. \\
& 2 \cdot k! \\
< \quad & \{k+1 > 2 \text{ since } k \geq 4\}. \\
& (k+1) \cdot k! \\
= \quad & \{\text{def'n of factorial}\} \\
& (k+1)!
\end{aligned}
$$

So, by transitivity of $<$, $2^{k+1} < (k+1)!$.

Since we showed $P(4)$ and $P(k) \to P(k+1)$ for all $k \geq 4$, by induction $P(n)$ is true for all $n \geq 4$.

Strong induction.



This is still OK — as long as we can prove each $P(k)$ from all the previous $P(j)$, not just $P(k-1)$.

More formally:

$$P(0) \wedge \left(\forall k:\mathbb{N}.\ \underbrace{(P(0)\wedge P(1)\wedge \dots \wedge P(k))}\rightarrow P(k+1)\right) \longrightarrow \left(\forall n:\mathbb{N}.\ P(n)\right).$$

↳ get to assume P is true for all previous numbers, not just $P(k)$.

(Principle of Strong Induction).

Note:

- Strong induction is actually logically equivalent to weak induction! But it can be easier to use.
- Base case could be something other than 0.
- Sometimes we need more than one base case to "get off the ground".

Example: Recall the Fibonacci numbers:

$(0, 1, 1, 2, 3, 5, 8, 13, 21, \dots)$

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Q: how fast do these grow?

Theorem. $F_n \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Failed proof attempt (w/ weak induction)

Let $P(n) \equiv (F_n \leq 2^{n-1})$. We will prove this for all n by induction.

Base case: $P(0) \equiv (F_0 \leq 2^{0-1}) \equiv (0 \leq \frac{1}{2})$ ✓.

Induction step: let k be an arbitrary natural # and suppose $P(k)$, that is,

$\boxed{F_k \leq 2^{k-1}.}$ KNOW

Then we must show $P(k+1)$, that is, $\boxed{F_{k+1} \leq 2^k.}$ WANT.

F_{k+1}

$=$ { def'n of Fibonacci #s } *

$F_k + F_{k-1}$

$\leq$ { induction hypothesis }

$(2^{k-1}) + F_{k-1}$

Stuck! We don't know anything about F_{k-1}.

* Also, $F_{k+1} = F_k + F_{k-1}$ is only valid if $k+1 \geq 2$, but it is not necessarily! Since we need to use 2 previous values, we must start with 2 base cases to "get off the ground".

Actual proof

By strong induction on n.

$P(0)$ is true as before: $F_0 = 0 \leq \frac{1}{2} = 2^{0-1}$. ✓

$P(1)$ says $F_1 \leq 2^{1-1}$, i.e. $1 \leq 1$ ✓.

In the induction step, let k be an arbitrary natural number ≥ 1. Suppose $P(0), P(1), P(2), \ldots, P(k)$ are all true. [Alt: Suppose $P(j)$ is true for all $j \leq k$.] Then we must show $P(k+1)$.

F_{k+1}

$=$ { def'n of Fibonacci, $k+1 \geq 2$ }.

$F_k + F_{k-1}$

$\leq$ { supposed $P(k)$ and $P(k-1)$ }.

$2^{k-1} + \underline{2^{k-2}}$

$<$ { $2^{k-2} < 2^{k-1}$ }

$\underline{2^{k-1}} + \underline{2^{k-1}}$

$=$ { algebra }

$\underline{2 \cdot 2^{k-1}}$

$=$ { algebra }.

2^k

if $a \leq b$ then $a + c \leq b + c$

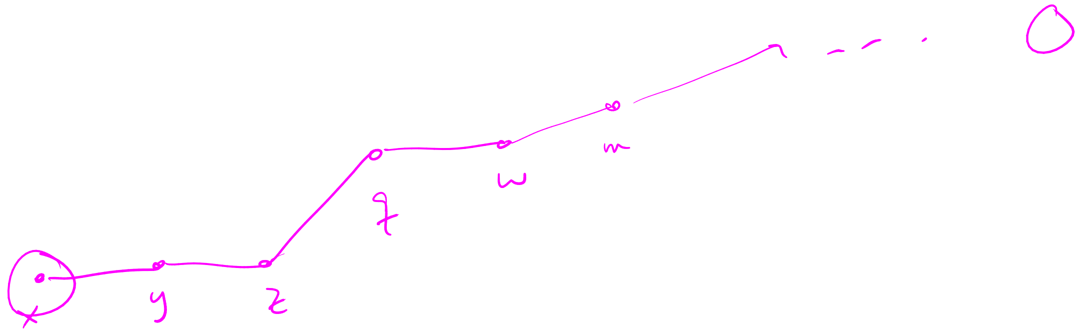
if $a \leq b$ then $ac \leq bc$ (if $c \geq 0$).

Therefore $F_{k+1} \leq 2^k$, which is $P(k+1)$.

So, by strong induction, $F_n \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

[Challenge: prove that $F_n \geq (3/2)^n$ for suitable n.]

$x = y = z < q = w \leq m < \ldots$



Ex. $a_0 = 0$

$a_n = 2a_{n-1} + 1.$

Prove that $a_n = 2^n - 1$ for all $n \geq 0$.

Proof. Let $P(n)$ be the proposition "$a_n = 2^n - 1$". We will prove $\forall n:\mathbb{N}.\ P(n)$ by induction. That is, we must show

$$P(0) \wedge [\forall k:\mathbb{N}.\ P(k) \longrightarrow P(k+1)].$$

Base case: $P(0)$ says "$a_0 = 2^0 - 1$" which is true since $a_0 = 0$ by definition and $2^0 - 1 = 1 - 1 = 0$.

Induction step: Let k be an arbitrary natural number, and suppose $P(k)$, that is, $\boxed{a_k = 2^k - 1.}$ KNOW We must show $P(k+1)$, that is, $\boxed{a_{k+1} = 2^{k+1} - 1.}$ WANT.

$$
\begin{aligned}
& a_{k+1} \\
= \quad & \{\text{definition of } a_{k+1}\} \\
& 2a_k + 1 \\
= \quad & \{\text{induction hypothesis}\} \\
& 2(2^k - 1) + 1 \\
= \quad & \{\text{algebra}\} \\
& 2^{k+1} - 2 + 1 \\
= \quad & \{\text{algebra}\}. \\
& 2^{k+1} - 1.
\end{aligned}
$$

✓

Ex. Prove that the sum of the Fibonacci numbers

$$F_0 + \dots + F_n = F_{n+2} - 1.$$

eg. $0, 1, 1, 2, 3, 5, 8, 13, 21$

$$\underbrace{0, 1, 1, 2, 3, 5}_{1+1+2+3+5=12} \qquad = 13 - 1.$$

Proof By induction.

Base case: when $n = 0$, $\underbrace{F_0}_{0} = \underbrace{F_{0+2} - 1}_{1-1=0}$ ✓

Ind. step. Let k be arbitrary natural #, and suppose

$$\boxed{F_0 + \dots + F_k = F_{k+2} - 1.}$$ KNOW

Then we must show

$$\boxed{F_0 + \dots + F_{k+1} = F_{k+3} - 1.}$$ WANT.

$$
\begin{aligned}
& F_0 + \dots + F_{k+1} \\
= \quad & \{\text{algebra}\} \\
& (\underline{F_0 + \dots + F_k}) + F_{k+1} \\
= \quad & \{\text{ind. hyp.}\} \\
& (\underline{F_{k+2}} - 1) + \underline{F_{k+1}} \\
= \quad & \{\text{def'n of Fibonacci}\}. \\
& F_{k+3} - 1
\end{aligned}
$$

Ex. Every natural number can be written as a sum of distinct powers of 2.

e.g. $5 = 2^0 + 2^2$

$11 = 2^0 + 2^1 + 2^3$

(This is binary! e.g. $5 = 101_2$, $11 = 1011_2$)

Proof. Let $B(n)$ be the proposition "n can be written as a sum of distinct powers of 2". We will prove $\forall n : \mathbb{N}.\ B(n)$ by strong induction.

Base case. $B(0)$ says 0 can be written as a sum of distinct powers of 2, which is true since we can add up no powers of 2 to get 0.

Induction step. Let $k \geq 1$ be an arbitrary natural number, and suppose $B(j)$ is true for all $j < k$. We must show $B(k)$ is true as well.

k must be either even or odd.

If k is even, then $k = 2m$ for some m. $m < k$, so we know $B(m)$ is true, ie.

$$m = 2^a + 2^b + 2^c + \dots$$

Then

$$k = 2m = 2(2^a + 2^b + 2^c + \dots)$$
$$= 2^{a+1} + 2^{b+1} + 2^{c+1} + \dots$$

So k can indeed be written as a sum of distinct powers of 2.

If k is odd, $k = 2m+1$. Again, since $m < k$, we know

$$m = 2^a + 2^b + 2^c + \dots$$

So

$$2m+1 = 1 + 2(2^a + 2^b + 2^c + \dots)$$
$$= 2^0 + \underbrace{2^{a+1} + 2^{b+1} + 2^{c+1}} + \dots$$

So $k = 2m+1$ can be written as a sum of <u>distinct</u> powers of 2 (note $a+1, b+1, \dots$ are all different, and none of them can be 0).