Sequences

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{5}, \ldots$

$5, 5, 5, 5, 5, \ldots\ 5, \ldots$

hello worl d

→ $2, 5, 8, 11, 14, 17, \ldots\ 20, \ldots$

→ $2, 6, 18, 54, 162, \ldots\ 486, \ldots$

→ $0, 1, 3, 7, 15, 31, 63, \ldots$

$1, 3, 5, 7, 9, \ldots$

$0, 1, 3, 6, 10, 15, \ldots$

Def'n A sequence is a function from $\mathbb{N}$ (or a subset of $\mathbb{N}$) to some set A. We typically use subscript notation like a_n to denote the nth term of the sequence, i.e. the output of the function for n.

In other words, a sequence is a list where each element has a number (or index):

$$a_0, a_1, a_2, a_3, \ldots$$

Occasionally we use $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ as indices.

eg. The sequence starting at index 1 with $a_n = \frac{1}{n}$.

$$1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots$$

eg. The sequence with $a_n = 5$.

$$5, 5, 5, 5, \ldots$$

eg. The finite sequence of characters

$a_0 = h, \quad a_1 = e, \quad a_2 = l, \quad a_3 = l, \quad a_4 = o, \quad a_5 = \text{'␣'}, \ldots$

eg. $2, 5, 8, 11, 14, 17, 20, \ldots$

Each term is 3 more than the previous.

$a_n = 3n + 2 \quad (n \geq 0)$

This kind of sequence, with a common difference between consecutive terms (ie. we add the same thing every time) is called an arithmetic sequence.

eg. $2, 6, 18, 54, 162, 486, \ldots$

Each term is $3\times$ the previous.

$\rightarrow \underline{a_n} = \underline{3a_{n-1}}$; $a_0 = 2$.] recurrence

$\rightarrow \underline{a_n = 2\cdot(3^n)} \quad (n \geq 0)$] closed form

This type of sequence, where we always multiply by the same number to get the next term (ie. "common ratio") is called a geometric sequence.

When we define a_n directly in terms of n, it is called a closed form.

Closed forms are nice:
- Directly calculate any term w/o knowing previous terms
- Tells us how quickly the sequence grows

BUT:
- Often more natural to describe sequences by recurrence
- Closed form might not exist, or might be complicated.

Def'n A recurrence for a sequence is a rule expressing a_n in terms of one or more previous terms.

eg: $a_n = 3a_{n-1}$

"The nth term is 3 times the previous $(n-1)^{st}$ term"

By itself, this describes many sequences, e.g.

1, 3, 9, 27, ...

2, 6, 18, 54, ...

0, 0, 0, 0, ...

79, ...

we typically pair it with a "base case" defining a_0 (or a_1), which uniquely defines the sequence.

eg: $\boxed{a_n = 2a_{n-1} + 1}$, $a_0 = 0$.

0, 1, 3, 7, 15, 31, ...

Conjectured closed form: $\boxed{a_n = 2^n - 1}$?

- Check it works for a_0? $a_0 = 0 = 2^0 - 1$ ✓
- See if a_n defined this way satisfies the recurrence?

  Substitute the conjectured closed form into the recurrence:

$$(2^n - 1) \stackrel{?}{=} 2\cdot(2^{n-1} - 1) + 1$$
$$= 2\cdot 2^{n-1} - 2 + 1$$
$$= 2^n - 1 \quad ✓$$

Examples.

$1, 3, 5, 7, 9, 11, \ldots$ (odd numbers)

$a_n = 2n + 1 \quad (n \geq 0)$ (closed form)

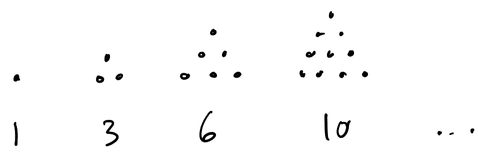
or

$$\left.\begin{array}{l} a_0 = 1 \\ a_n = a_{n-1} + 2 \end{array}\right\} \text{ recurrence}$$

next term previous term

$0, 1, 3, 6, 10, 15, \ldots$ (Triangular numbers)

$a_0 = 0$
$a_n = a_{n-1} + n$



$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (Fibonacci numbers)

Each number is sum of previous two.

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$