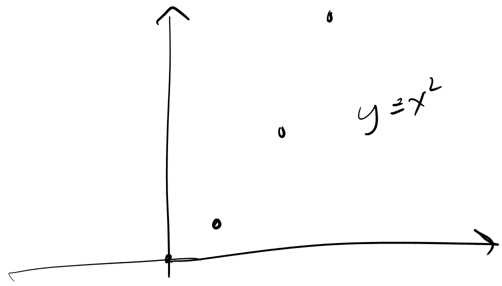
Functions

What is a function?

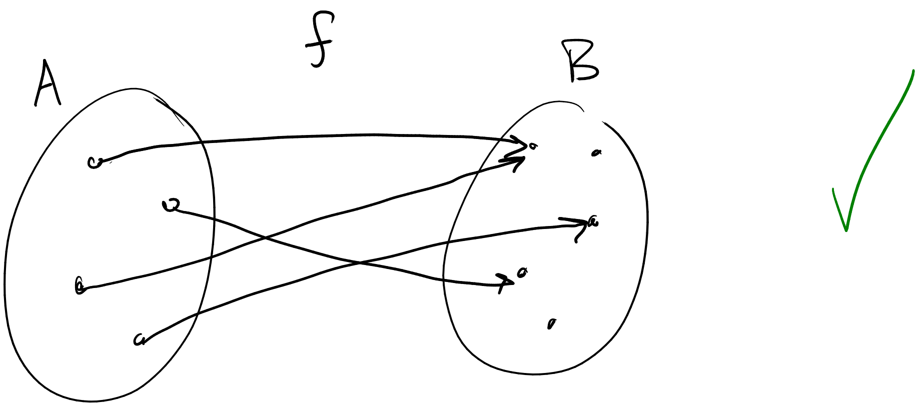
input → operation → output

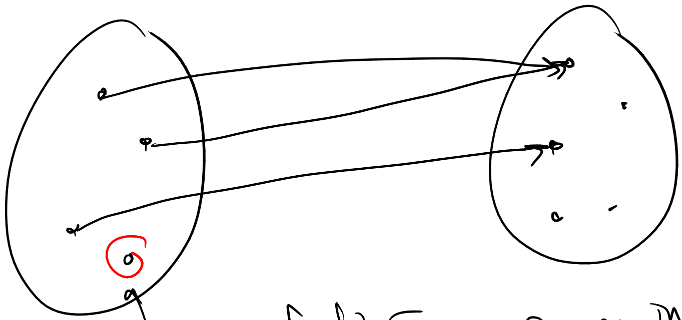
graphable – 2D plot



Def'n Informally, a function is a rule that associates outputs with inputs. Formally, let A and B be sets (types). A function f from A to B, written $f : A \to B$, is a relation from A to B such that every $a \in A$ is related to exactly one $b \in B$.

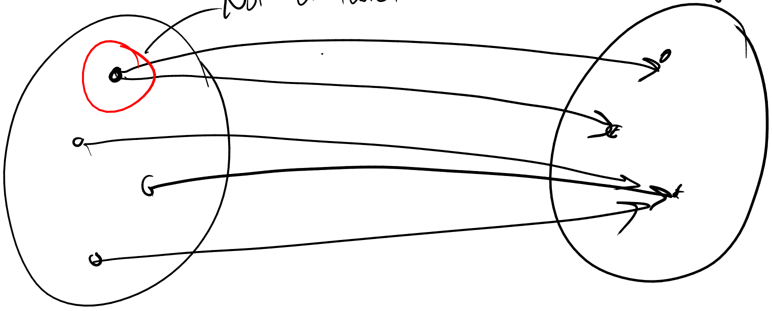
A is called the domain and B is called the codomain. We write $f(a) = b$ to denote the unique $b \in B$ associated with $a \in A$.

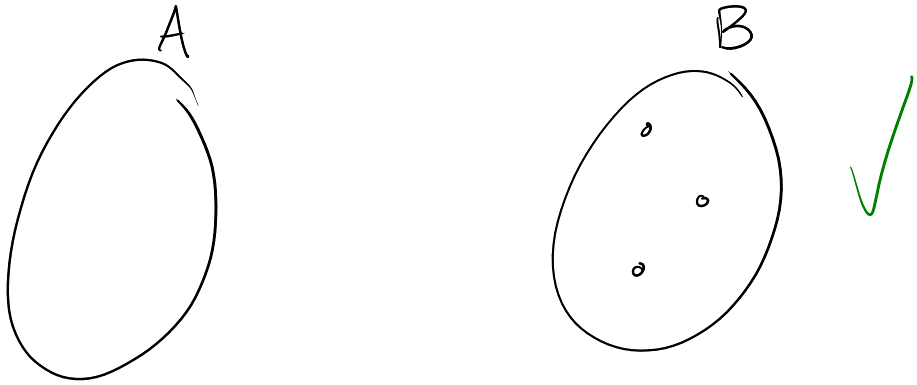




Not a function – must be defined for every input.

Not a function — 2 different outputs for 1 input.



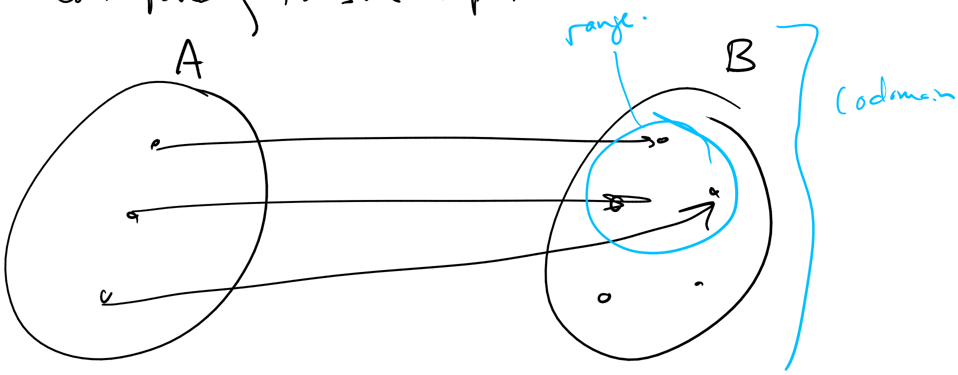


"empty function"

Defn The range of a function $f : A \to B$ is the set

$$\{ b \mid b \in B,\ \exists a : A .\ f(a) = b \}$$

ie. the subset of B consisting of all b's which are actually outputs corresponding to some input.



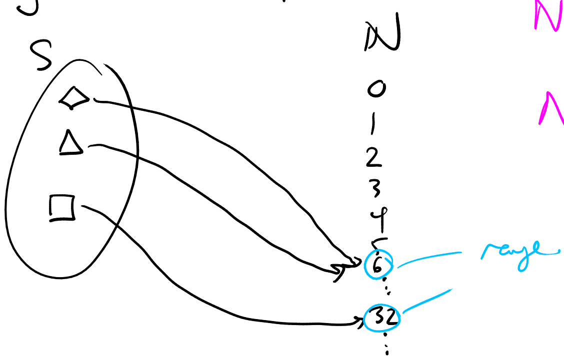
codomain = potential outputs

range = actual outputs

---

ex. $S = \{ \diamond, \Delta, \square \}$. Define $g : S \to \mathbb{N}$ by

$$g(\diamond) = 6$$
$$g(\Delta) = 6$$
$$g(\square) = 32$$



Not 1-1.

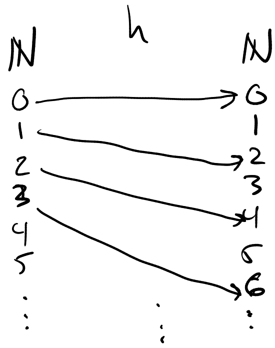
Not onto.

ex. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by

$$h(n) = 2n.$$

domain = codomain = $\mathbb{N}$

range = even natural #'s.

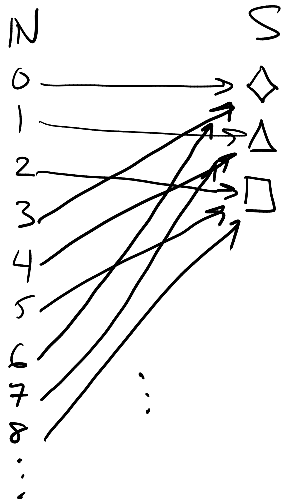


1-1 ✓

Not onto.

eg. $\ell : \mathbb{N} \to S$ be the function which sends all multiples of 3 to $\diamond$, all one more than mult. of 3 to Δ, and 2 more than mult. of 3 to $\square$

$$\ell(3n) = \diamond$$
$$\ell(3n+1) = \Delta$$
$$\ell(3n+2) = \square.$$
~~$\ell(6) = \square$.~~



Not 1-1

Onto ✓

ex. $j : \mathbb{N} \to \mathbb{N}$ defined by

$$j(3n) = 2n+5$$
$$j(3n+1) = n^2$$
$$j(3n+2) = 17.$$

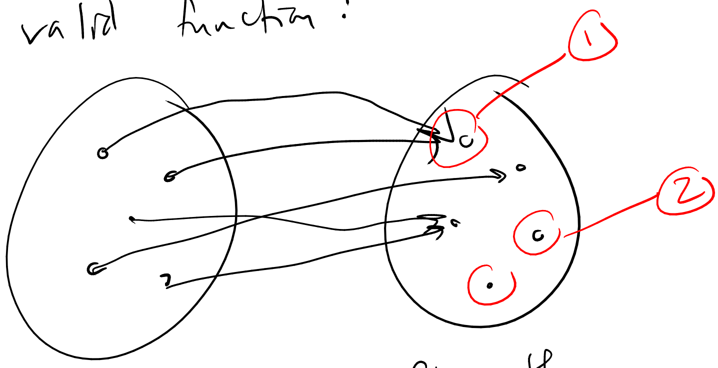
Not 1-1

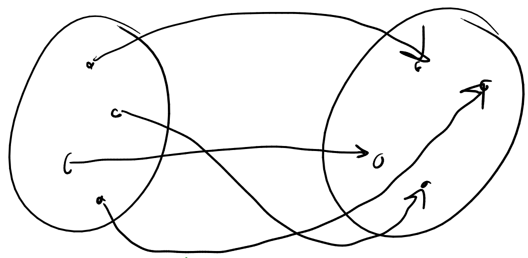
Not onto.

eg. $j(6) = j(3 \cdot 2) = 2 \cdot 2 + 5 = 9.$

$j(8) = 17.$

Motivation: What if we want to "turn a function around" into a function that does the opposite? Needs to be a valid function!





✗ Can't flip the arrows — result would not be a valid function.

✓ Can flip the arrows!

---

1 **Def'n** A function $f: A \to B$ is <u>one-to-one</u> (<u>injective</u>, <u>an injection</u>) iff no two elements of the domain A map to the same element of the codomain B.

That is,

$$\forall a_1, a_2 \in A.\ a_1 \neq a_2 \longrightarrow f(a_1) \neq f(a_2).$$

Or, equivalently, taking the contrapositive,

$$\forall a_1, a_2 \in A.\ f(a_1) = f(a_2) \longrightarrow a_1 = a_2. \quad *$$

(We typically use the 2nd version for proving a function injective: by supposing $f(a_1) = f(a_2)$, and showing $a_1 = a_2$.

How would we show a function is not injective?

$$\neg \left(\forall a_1, a_2 \in A.\ f(a_1) = f(a_2) \rightarrow a_1 = a_2 \right)$$

{de Morgan}

$$\equiv \exists a_1, a_2 \in A.\ \neg \left(f(a_1) = f(a_2) \rightarrow a_1 = a_2 \right)$$

$\{P \rightarrow Q \equiv \neg P \vee Q\}$

$$\equiv \exists a_1, a_2 \in A.\ \neg \left(f(a_1) \neq f(a_2) \vee a_1 = a_2 \right)$$

{de Morgan}

$$\equiv \exists a_1, a_2 \in A.\ f(a_1) = f(a_2) \wedge a_1 \neq a_2 .$$

ex. • $g : S \rightarrow \mathbb{N}$ is not one-to-one since $g(\triangle) = g(\diamond) = 6$.

• $h : \mathbb{N} \rightarrow \mathbb{N}$
$h(n) = 2n$ is injective. Proof:

Let $x, y \in \mathbb{N}$ be arbitrary. Suppose $h(x) = h(y)$.

$h(x) = h(y)$

$\rightarrow$ {def'n of h}

$2x = 2y$

$\rightarrow$ {algebra}.

$x = y$.

②

Def'n A function $f: A \to B$ is onto (surjective, a surjection) if every element of B is an output for some input, that is,

$$\forall b \in B.\ \exists a \in A.\ f(a) = b.$$

Put another way, f is onto if the range is the entire codomain.

Proving f is not onto?

$$\neg(\forall b \in B.\ \exists a \in A.\ f(a) = b)$$

$$\equiv \quad \{\text{de Morgan, twice}\}.$$

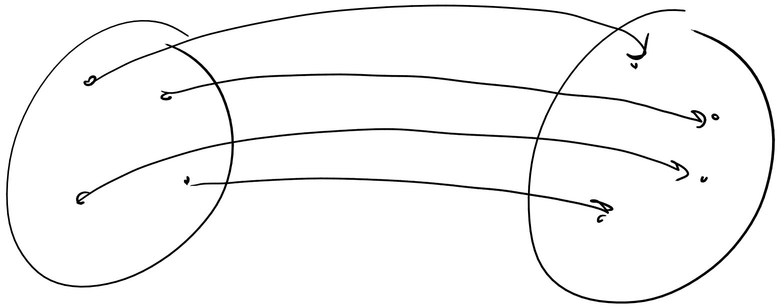
$$\exists b \in B.\ \forall a \in A.\ f(a) \neq b.$$

Def'n A function $f: A \to B$ which is both one-to-one and onto is called invertible (bijective, a bijection).

ex. $f: \mathbb{N} \to \mathbb{N}$
$f(n) = n$

$h: \mathbb{Q} \to \mathbb{Q}$
$h(x) = mx + b$

$g: \mathbb{Z} \to \mathbb{Z}$
$g(n) = -n$

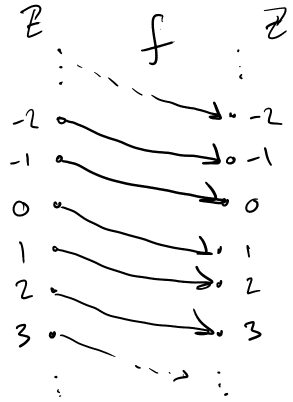


A bijection pairs up the elements of two sets.

<u>Notation</u>. We write f^{-1} ("f inverse") for the inverse function of f, that is $(f^{-1}(b) = a) \leftrightarrow (f(a) = b)$.

<u>Note</u>: Usually $x^{-1} = \frac{1}{x}$. But $f^{-1} \neq \frac{1}{f}$.

<u>Ex</u>. Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$. Is f invertible?



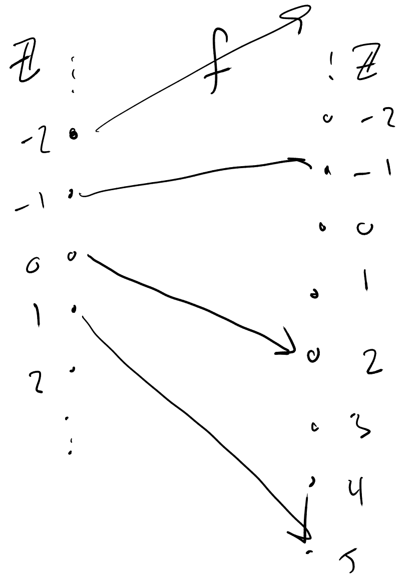
Yes! 1-1 and onto.

And $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is $f^{-1}(y) = y - 1$

<u>Ex</u>. $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \lfloor n/2 \rfloor$. Is this invertible?

No, not 1-1.

<u>Ex</u>. $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 2$. Is it invertible?



No, not onto.

Ex. $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3x + 2$.

Let's try proving it is 1-1 and onto!

- 1-1. We must show $\forall x, y : \mathbb{Q}.\ (f(x) = f(y)) \to (x = y)$.

  Let $x, y \in \mathbb{Q}$ be arbitrary. Suppose $f(x) = f(y)$.

$$
\begin{aligned}
& f(x) = f(y) \\
\to\ & 3x + 2 = 3y + 2 && \{\text{def'n of } f\} \\
\to\ & 3x = 3y && \{\text{algebra}\}. \\
\to\ & x = y\ \checkmark && \{\text{algebra}\}.
\end{aligned}
$$

- onto. We must show $\forall y : \mathbb{Q}.\ \exists x : \mathbb{Q}.\ f(x) = y$.

  So let $y \in \mathbb{Q}$ be arbitrary. We want to pick x such that $f(x) = y$, that is, $3x + 2 = y$.

$$
\begin{aligned}
& 3x + 2 = y \\
\leftrightarrow\ & 3x = y - 2 \\
\leftrightarrow\ & x = \frac{y-2}{3}
\end{aligned}
$$

So yes, we can pick $x = \frac{y-2}{3}$, which is a rational # since rationals are closed under $-$, $\div$.

$$f^{-1}(y) = \frac{y-2}{3}.$$