Relations

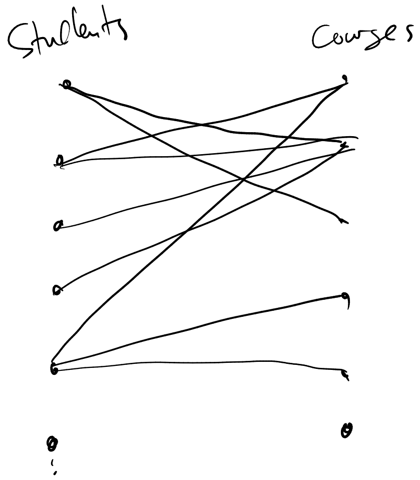
Def'n Let A and B be sets. A *relation* from A to B is a subset of the Cartesian product $A \times B$.

We write $a R b$ to denote $(a, b) \in R$ ("a is related to b by R").

Example Let S be the set of students @ Hendrix
C the set of courses offered next year.

Let R be the relation

$$R = \{(s, c) \mid s \text{ is preregistered for } c\}.$$



Def'n A *relation on a set* S is a relation from S to S (i.e., a subset of $S \times S$).

Examples:

First, some relations on $\mathbb{N}$:

	Refl	Sym	Trans.
$R_1 = \{(a, b) \mid a \leq b\}$.	✓		✓
$R_2 = \{(a, b) \mid a > b\}$.			✓
$R_3 = \{(a, b) \mid a = b\}$.	✓	✓	✓

- $R_4 = \{(a,b) \mid a = b+1\}$.
- $R_5 = \{(a,b) \mid a+b \leq 10\}$ ✓
- $R_6 = \{(a,b) \mid a \text{ evenly divides } b\}$ ✓ ✓
- $R_7 = \{(a,b) \mid a \text{ and } b$ end with the same digit in base 10$\}$. ✓ ✓ ✓

And a few more:

- $R_8 = \{(P,Q) \mid P, Q \text{ propositions}, P \to Q\}$. ✓ ✓
- $R_9 = \{(P,Q) \mid P \leftrightarrow Q\}$. ✓ ✓ ✓
- $R_{10} = \{(S,T) \mid S \subseteq T\}$. ✓ ✓

Relations can have various nice properties.

① Reflexivity

Def'n A relation R on a set A is reflexive if

$$\forall a \in A.\ a \, R \, a.$$

that is, every element is related to itself.

② Symmetry

Defn A relation R on a set A is symmetric if

$$\forall a, b \in A.\ (a \, R \, b) \to (b \, R \, a).$$

(i.e. order does not matter.)

③ Transitivity

Def'n A relation R on a set A is *transitive* if

$$\forall a, b, c \in A . (a \, R \, b) \wedge (b \, R \, c) \longrightarrow (a \, R \, c)$$

Def'n An *equivalence relation* (or just "equivalence") is a reflexive, symmetric, transitive relation

Examples

a) The equality relation $=$ itself.

b) Consider the relation L on the set of all English words, where two words are related if & only if they have the same number of letters.

eg dog L cat.

reflexive? ✓ every word has same length as itself.

symmetric? ✓ yes, order of words doesn't matter.

transitive? ✓ a, b same length

b, c same length $\longrightarrow$ a, c same length.

c) Last digit same (R_7 from examples before).

d) $\longleftrightarrow$

e) $\{(a,b) \mid |a-b| \leq 1\}$?

reflexive ✓

symmetric ✓

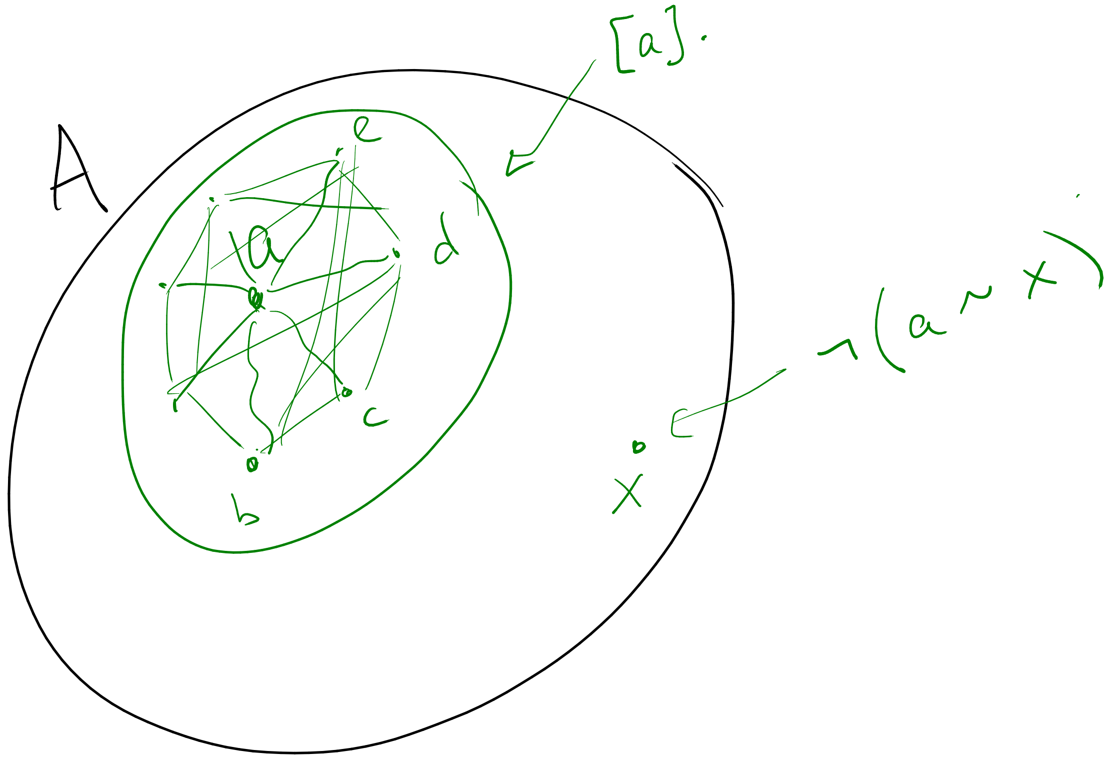
but <u>not</u> transitive — δ.

$(10,11)$ $(11,12)$ are in relation

but $(10,12)$ is not.

<u>Def'n</u> Let $\sim$ be an equivalence relation on A, and let $a \in A$. The <u>equivalence class</u> of a, written $[a]$, is the set of all elements related to a, i.e.

$$[a] = \{ b \mid b \in A,\ a \sim b \}.$$



<u>ex.</u>

L, words by length

$$[\text{dog}] = \{\text{dog}, \text{cat}, \text{cow}, \text{hat}, \text{eat}, \ldots\}$$

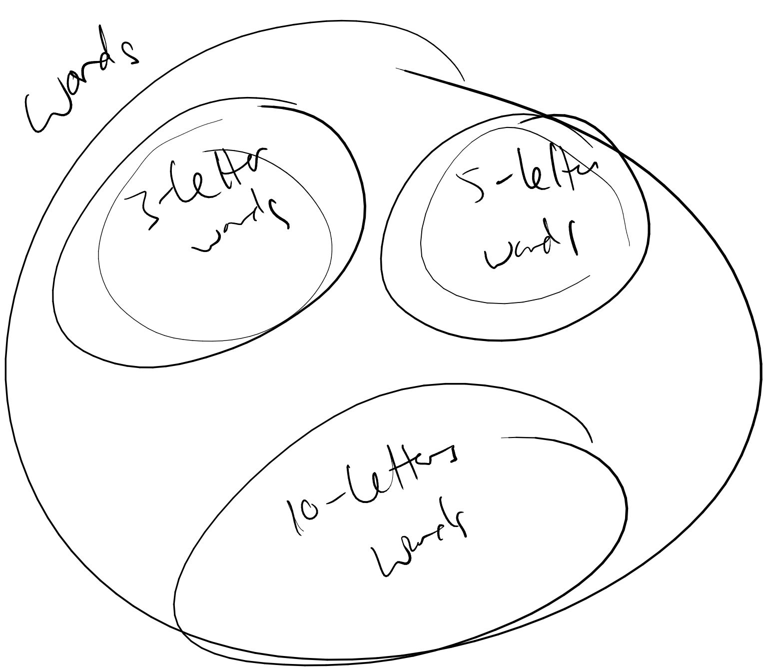
$$[\text{a}] = \{\text{a}, \text{I}, \text{O}\}$$

$$[\text{stationary}] = \{\ldots \text{ten letter words} \ldots\}$$

ex. R_7, numbers w/ same last digit.

$$[1] = \{\underline{1, 11, 21, 31, \ldots 561, \ldots}\} = [561]$$

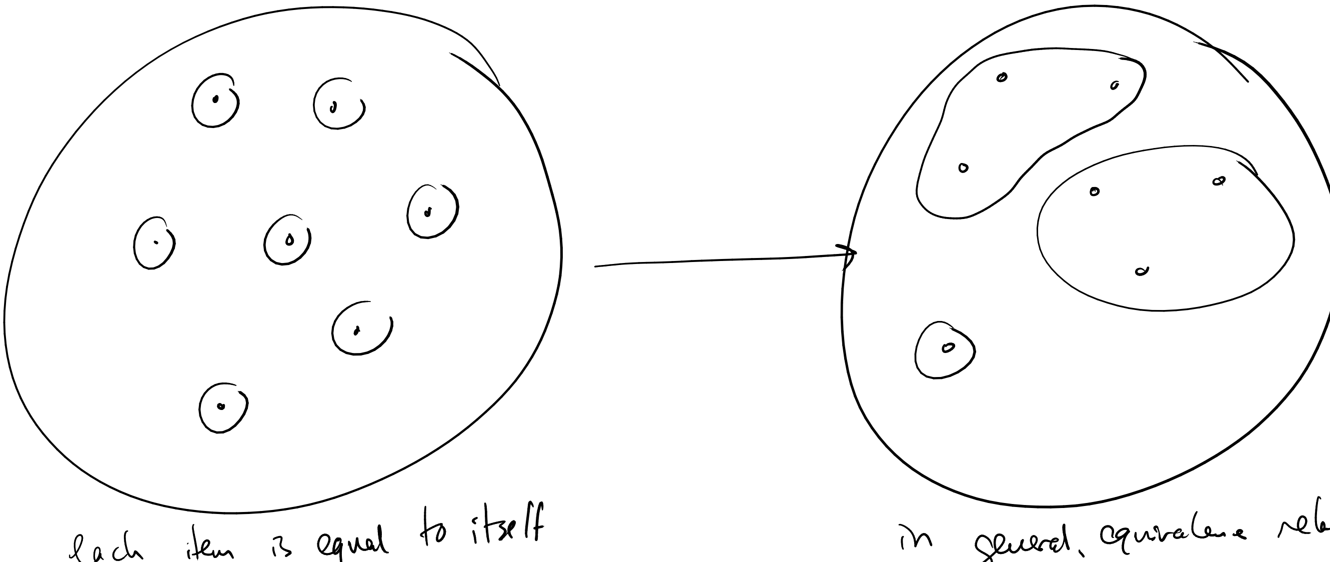
$$[1{,}373{,}323] = \{3, 13, 23, \ldots\}$$



Review:

- Properties of equivalence relation?
- Is $\leq$ an equivalence relation on $\mathbb{N}$?

$=$ relation

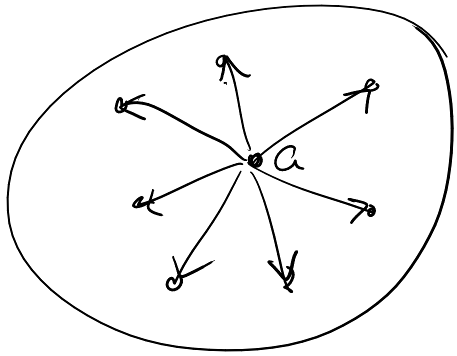


each item is equal to itself and nothing else

in general, equivalence relation means we don't care about certain things, i.e. want to consider some things as "the same".

Claim: equivalence classes "chop up" (partition) the set into separate blobs (subsets) of things that are equivalent.

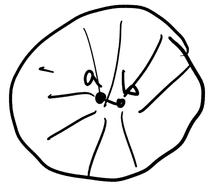
Reminder: $[a] = \{ x \mid a \sim x \}$.



Theorem. Let $\sim$ be an equivalence relation on a set A. For all $a, b \in A$, the following three propositions are all logically equivalent:

① $a \sim b$

② $[a] = [b]$



③ $[a] \cap [b] \neq \varnothing$

We will prove

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}$$

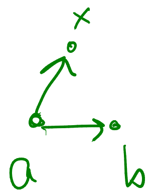
. Then by transitivity of $\rightarrow$, they are all $\leftrightarrow$.

Proof. Let $a, b \in A$ be arbitrary.

- ① → ②, i.e. $(a \sim b) \to ([a] = [b])$. So

Suppose $\boxed{a \sim b}$. We will show $\underline{[a] \subseteq [b]}$ and $\underline{[b] \subseteq [a]}$.

To show $[a] \subseteq [b]$, suppose $\boxed{x \in [a]}$; we must show $\boxed{x \in [b]}$. Since $x \in [a]$, by definition $\boxed{a \sim x}$.



By symmetry, $b \sim a$.

Then by transitivity, $b \sim x$.

So by definition, $\underline{x \in [b]}$.

$[b] \subseteq [a]$ is exactly parallel to previous argument.

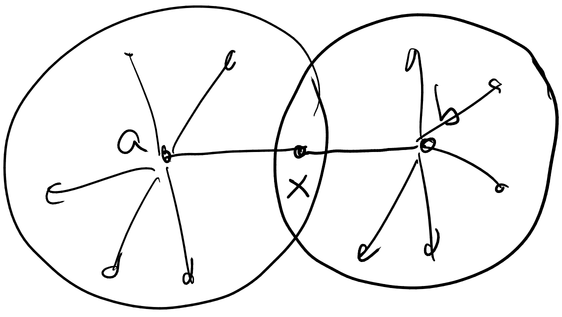
- ② → ③, i.e. $([a] = [b]) \to ([a] \cap [b] \neq \emptyset)$.

$$
\begin{aligned}
& [a] \cap [b] \\
= \quad & \{ [a] = [b] \} \\
& [a] \cap [a] \\
= \quad & \{ \cap \text{ is idempotent} \}. \\
& [a]
\end{aligned}
$$

So is $[a] \neq \emptyset$?

By reflexivity, $a \sim a$, hence $a \in [a]$, so $[a] \neq \emptyset$.

- ③ $\rightarrow$ ①, i.e. $([a] \cap [b] \neq \emptyset) \rightarrow (a \sim b)$.



Suppose $[a] \cap [b] \neq \emptyset$, then there must be at least one $x \in [a] \cap [b]$; which by definition means $x \in [a] \wedge x \in [b]$. Hence, $a \sim x$ and $b \sim x$. So, by symmetry and transitivity, $a \sim b$.

Hence, equivalence classes partition a set:

- every element is in some equivalence class (by reflexivity)

* No two equivalence classes overlap.