Theorem - Set union is idempotent, that is, $A \cup A = A$ for all sets A.

Proof. Let A be an arbitrary set. We will show $A \cup A = A$, by showing $A \cup A \subseteq A$ and $A \subseteq A \cup A$.

[$A \cup A \subseteq A$]. Let $x \in A \cup A$, we must show $x \in A$.

$x \in A \cup A$ means $x \in A \vee x \in A \equiv x \in A$ since $\vee$ is idempotent.

[$A \subseteq A \cup A$]. Let $x \in A$, then $x \in A \vee x \in A$ since $\vee$ is idempotent, which means $x \in A \cup A$.

Theorem For all sets S and T,

$$\overline{S \cup T} = \overline{S} \cap \overline{T}.$$

Proof. Let S and T be arbitrary sets in some universe $\mathcal{U}$. We will show $\overline{S \cup T} = \overline{S} \cap \overline{T}$ by showing subset in both directions.

[$\overline{S \cup T} \subseteq \overline{S} \cap \overline{T}$]. Let $x \in \overline{S \cup T}$; we will show $x \in \overline{S} \cap \overline{T}$.

(arbitrary)

$$
\begin{aligned}
& x \in \overline{S \cup T} \\
\equiv \quad & \{\text{definition of complement}\} \\
& \neg (x \in S \cup T) \\
\equiv \quad & \{\text{def'n of union}\} \\
& \neg (x \in S \vee x \in T) \\
\equiv \quad & \{\text{de Morgan}\} \\
& \neg (x \in S) \wedge \neg (x \in T) \\
\equiv \quad & \{\text{defn of complement}\} \\
& x \in \overline{S} \wedge x \in \overline{T} \\
\equiv \quad & \{\text{def'n of intersection}\} \\
& x \in \overline{S} \cap \overline{T}.
\end{aligned}
$$

[$\overline{S} \cap \overline{T} \subseteq \overline{S \cup T}$]. Let $x \in \overline{S} \cap \overline{T}$, we will show $x \in \overline{S \cup T}$.
We already proved above that $(x \in \overline{S \cup T}) \equiv (x \in \overline{S} \cap \overline{T})$.

[In fact, in general, to prove $A = B$, we can also prove $\underline{\forall x : U.\ (x \in A) \leftrightarrow (x \in B)}$.]

<u>Theorem</u>. For all sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

<u>Proof</u>. Let A, B, C be arbitrary, and let x be arbitrary. We will show

$$(x \in A \cap (B \cup C)) \equiv (x \in \underline{(A \cap B) \cup (A \cap C)}).$$

$x \in A \cap (B \cup C)$

$\equiv \qquad \{\ ?\ \}$

? exercise.

Theorem. For all sets A and B,

$$(A \subseteq B) \longleftrightarrow (A \cup B = B).$$

Proof Let A and B be arbitrary sets. We will prove both directions of the if + only if.

$[(A \subseteq B) \rightarrow (A \cup B = B)]$.