Algebraic operations on sets.

① Set Cardinality (size)

Def'n The cardinality of a finite set is the number of distinct elements it contains (i.e. a natural number). The cardinality of a set A is written $|A|$.

e.g. $|\{1, 3, 5\}| = 3.$

$|\{2, 4, \dots, 100\}| = 50.$

$|\emptyset| = 0.$

② (Cartesian) product

Def'n. If A and B are sets, the (Cartesian) product of A and B, written $A \times B$, is the set of all possible ordered pairs of elements from A and B. That is,

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}.$$

"cross", "times"

e.g. $A = \{1,2,3\}$

$B = \{\Delta, \square\}$

then $A \times B = \{ (1,\Delta), (2,\square), (1,\square), (3,\square), (2,\Delta), (3,\Delta) \}$

	Δ	$\square$
1	$(1,\Delta)$	$(1,\square)$
2	$(2,\Delta)$	$(2,\square)$
3	$(3,\Delta)$	$(3,\square)$

We can think of $A \times B$ as filling in a 2D table where the rows are labelled with elements from A and columns with elements from B. Each cell corresponds to pair of row + column.

In general, this shows that

$$|A \times B| = |A| \cdot |B| .$$

③ Power set

Def'n The power set of a set A, written $P(A)$, is the set of all possible subsets of A.

e.g. Let $A = \{1, 2, 3\}$. Then

$$P(A) = \{\{1,2\}, \emptyset, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\} \quad (2^3 = 8)$$

e.g. $P(\{4,5\}) = \{\{4\}, \{5\}, \emptyset, \{4,5\}\}$ $\quad (2^2 = 4)$

$P(\{\square\}) = \{\{\square\}, \{\}\}$ $\quad (2^1 = 2)$

$P(\emptyset) = \{\emptyset\}$ $\quad (2^0 = 1)$

How to systematically list all elements of a subset?

eg $\{1, 2, 3\}$.

1?	2?	3?	subset
T	T	T	$\{1,2,3\}$ ←
T	T	F	$\{1,2\}$
T	F	T	$\{1,3\}$
T	F	F	$\{1\}$
F	T	T	$\{2,3\}$
F	T	F	$\{2\}$
F	F	T	$\{3\}$
F	F	F	$\emptyset$ ←

In general, if A is a finite set,

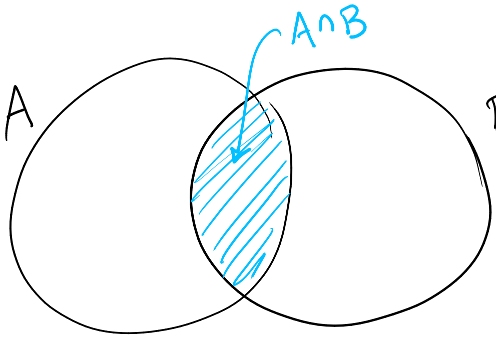
$$|P(A)| = 2^{|A|} .$$

Because each element of A gives us 2 independent choices: whether to include it or not.

④ Intersection.

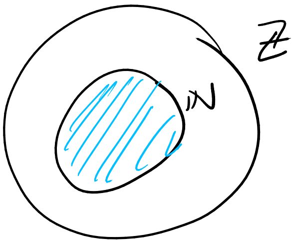
<u>Def'n</u> The <u>intersection</u> of sets A and B is the set of elements common to both:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



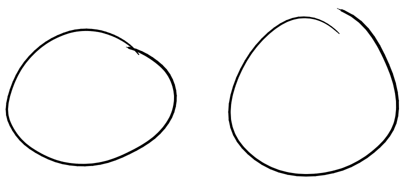
<u>eg.</u> $\{1, 3, 5, 2\} \cap \{3, \ldots, 10\} = \{3, 5\}$

<u>eg.</u> $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}.$



[Exercise: prove that $(A \cap B = A) \leftrightarrow (A \subseteq B)$]

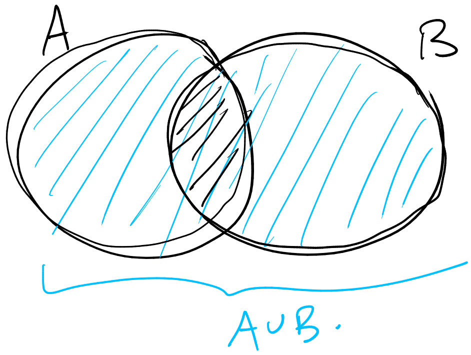
eg. $\{1/2, 3/4, 4/5\} \cap \mathbb{N} = \emptyset$



⑤ <u>Union</u>.

<u>Def'n</u> The <u>union</u> of sets A and B is the set of all elements which are in either one (or both):

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$



eg $\{1,2,4\} \cup \{3,4,5\} = \{1,2,3,4,5\}$.

eg. $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$

- $|A \cup B| \geq |A|$
- $|A \cup B| \geq |B|$
- $|A \cup B| = |A| + |B| - |A \cap B|$

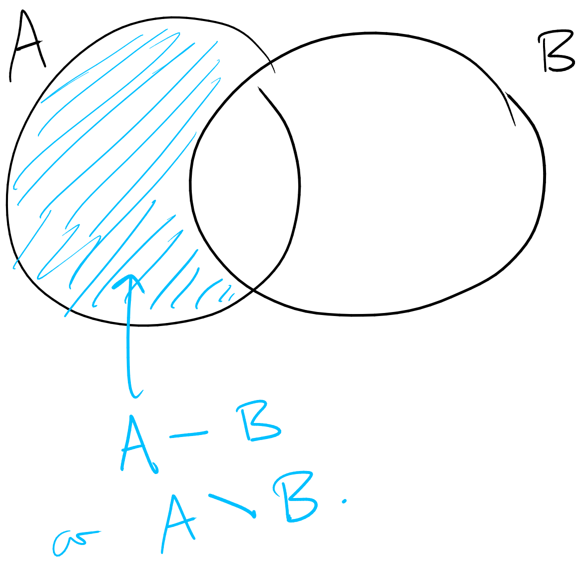
baby version of Principle of Inclusion-Exclusion (PIE)

- $|A \cup B| = |B|$ if $A = \emptyset$
- Also $\emptyset \cup B = B$ ($\emptyset$ is identity for $\cup$).

7) Difference

Def'n The difference of two sets A and B, written $A - B$ or $A \setminus B$, consists of the elements which are in A but not B.

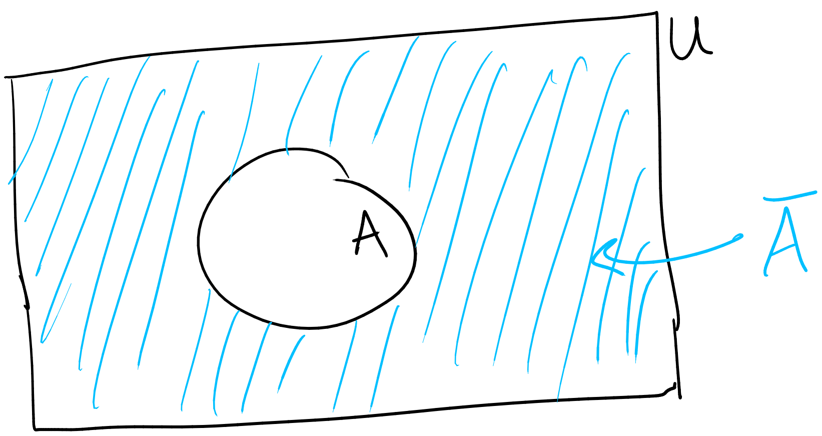
$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



⑧ Complement

<u>Def'n</u> Given a universal set U (i.e. universe of discourse), the <u>complement</u> of a set A is the set of all elements of U which are not in A:

$$\overline{A} = U - A = \{x \mid x \in U \wedge x \notin A\}.$$



e.g. if $U = \mathbb{N}$, $\overline{\{0,2,4,\ldots\}} = \{1,3,5,\ldots\}$

Algebraic laws for set operations

$\cap$, $\cup$, complement satisfy many of the same algebraic laws as $\wedge$, $\vee$, $\neg$. For example:

- $\cup$ is idempotent, i.e. $A \cup A = A$.
- $\cup$ is associative: $A \cup (B \cup C) = (A \cup B) \cup C$.
- $\cap$ distributes over $\cup$ and vice versa:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
- de Morgan laws:
$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$
- $\overline{\overline{A}} = A$