Set Theory

Def'n A set is an unordered, finite or infinite collection of objects, called elements or members of the set. Sets cannot contain a given element more than once, and the order of elements in a set does not matter. Put another way, the only thing that matters about a given element is whether it is in the set or not.

We typically use capital letters for sets and lowercase letters for elements.

$\in$ or "x is in S"

The notation $x \in S$ ("x is an element of S") is a proposition which says x is an element of S.

$x \notin S$ is an abbreviation for $\neg(x \in S)$.

Notation / examples.

$S = \{1, 3, 5, 9\}$. — set w/ 4 elements.

$S = \{1, 2, 3, \ldots, 100\}$ — $\ldots$ means "and so on".
this is set of 100 elements
all #'s from 1 to 100.

$\{cow, horse, dog\}$

$\{53\}$ — set with one element.

<u>Defn</u>

$\{\}$ or $\emptyset$ is the "empty set" which has zero elements.

Something like

$$\{4, \text{True}, \text{cow}, \text{"hello"}\}$$

is allowed by the definition but we will never use this — all our sets will have elements of the same type.

One more example:

$$\{\{1, 3\}, \{2, 5\}, \{7\}, \emptyset\}$$

Set builder notation

General form:

$$\{ \text{expression} \mid \text{variable definition(s), conditions} \}$$

elements

"such that"

Ex.

$$\{ x \mid x \in \mathbb{Z},\ 1 \leq x < 5 \}$$

$$= \{1, 2, 3, 4\}$$

Eg.

$$\{ x \mid x \in \mathbb{Z},\ \text{odd}(x),\ 0 \leq x < 100 \}$$

Multiple conditions (as many as you want!) = AND

$$= \{1, 3, 5, \ldots, 99\}$$

Eg.

$$\{ \underline{2x + 5} \mid x \in \mathbb{N},\ \underline{x \leq 10} \}$$

- x is nat. number ≤ 10
- For each such x, $2x+5$ will be a member of our set.

$$= \{5, 7, 9, \ldots, 25\}$$

eg. $\{ x^2 \mid x \in \mathbb{Z},\ \underline{-3 \le x \le 3} \}$

$= \{0, 1, 4, 9\}$

eg. $\{ \underbrace{(x,y)} \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y = 6 \}$

Gives <u>all possible combinations</u> of values for x, y where $x + y = 6$.

$= \{(6,0), (3,3), (1,5), (4,2), (10,-4), (0,6), \dots\}$

An infinite set!

<u>Some special sets</u>

- $\emptyset$, the empty set
- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the natural numbers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the integers.
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, positive integers.
- $\mathbb{Q} = \{ p/q \mid p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \}$
  (Actually, need something else to say eg $1/2 = 2/4$).
- $\mathbb{R}$ = real #'s
- $\mathbb{C}$ = complex #'s.

Subsets

Def'n A is a subset of B, written $A \subseteq B$, iff every element of A is also an element of B. That is,

$$(A \subseteq B) \equiv (\forall a \in A.\ a \in B).$$

eg.

$$\{1, 3, 6\} \subseteq \{2, 3, 7, 6, 1\}$$

$$\{1, 3, 6\} \subseteq \mathbb{N}$$

$$\{1, 3, 6\} \not\subseteq \{2, 7, 6, 1\}$$

Question

Is a set A a subset of itself? i.e. is $A \subseteq A$?

Answer: by definition, yes!

$$\forall a \in A.\ a \in A. \quad \checkmark$$

Question

Is the empty set a subset of $\mathbb{N}$?

By definition, $\emptyset \subseteq \mathbb{N}$ means

$$\forall a \in \emptyset.\ a \in \mathbb{N}.$$

What do we do w/ $\forall$ when the domain is empty?

Several arguments why this is true:

- Negation is $\neg(\forall a \in \emptyset .\ a \in \mathbb{N})$

$$\equiv \exists a \in \emptyset .\ a \notin \mathbb{N}.$$

Clearly false - there are no elements in $\emptyset$! Hence its negation should be true.

- $\forall a \in \emptyset .\ a \in \mathbb{N}$

$$\equiv \forall a \in \mathbb{N} .\ (a \in \emptyset) \to (a \in \mathbb{N}).$$

$$\equiv \forall a \in \mathbb{N} .\ F \to (a \in \mathbb{N}).$$

$$\equiv \forall a \in \mathbb{N} .\ T$$

$$\equiv T.$$

- Subset = what is left over after taking things out.

So, in fact, yes, $\emptyset \subseteq \mathbb{N}$.

More generally, $\emptyset \subseteq A$ for any set A.

Def'n Two sets are equal if each is a subset of the other:

$$(A = B) \equiv (A \subseteq B) \wedge (B \subseteq A).$$

Practically, we can use this to prove that two sets are equal — even when they have different-looking definitions.

Ex. Let $A = \{2x+3 \mid x \in \mathbb{Z}\}$

$B = \{x \mid x \in \mathbb{Z}, \text{Odd}(x)\}$.

Prove $A = B$.

Proof

By definition, $A = B$ means $(A \subseteq B) \wedge (B \subseteq A)$. So we will prove both.

We must show $A \subseteq B$, which means $\forall a \in A.\ a \in B$. So let f be an arbitrary element of A; we will show $f \in B$.

Since $f \in A$, $f = 2x+3$ for some $x \in \mathbb{Z}$. We know $f \in \mathbb{Z}$, and also $f = 2(x+1)+1$, so by definition f is odd, and hence $f \in B$.

Other half $(B \subseteq A)$ — exercise. (+ in lecture notes).