Algebraic Laws for Propositional Logic

2 organizing principles:

① Most equivalences/laws have an "opposite world" version where we switch $\wedge$ / $\vee$ and T/F.

② We can get some intuition by thinking in terms of 0/1:

- F ~ 0
- T ~ 1
- $\neg$ ~ subtracting from 1
- $\wedge$ ~ multiplication
- $\vee$ ~ addition (but w/ max of 1)

Name	Equivalence	"Opposite world" equivalence
Identity	$p \wedge T \equiv p$	$p \vee F \equiv p$
Annihilation	$p \wedge F \equiv F$	$p \vee T \equiv T$
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
Commutativity	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$.
Associativity	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
	$p \wedge \neg p \equiv F$ (contradiction)	$p \vee \neg p \equiv T$ (Law of Excluded Middle)

Also:

- Double negation elimination: $\neg(\neg p) \equiv p$
- Implication: $p \rightarrow q \equiv \neg p \vee q$
- Iff: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Example.

Simplify: $\neg(p \to \neg(\neg q \vee (r \wedge q)))$

$\equiv$ {distributivity}

$\neg(p \to \neg((\neg q \vee r) \wedge (\neg q \vee q)))$

$\equiv$ {excl. middle}

$\neg(p \to \neg((\neg q \vee r) \wedge \top))$

$\equiv$ {identity}

$\neg(p \to \neg(\neg q \vee r))$

$\equiv$ {de Morgan}

$\neg(p \to (\neg\neg q \wedge \neg r))$

$\equiv$ {double negation}

$\neg(p \to (q \wedge \neg r))$

$\equiv$ {implication}

$\neg(\neg p \vee (q \wedge \neg r))$

$\equiv$ {de Morgan, double negation}

$p \wedge \neg(q \wedge \neg r)$

{de M., double negation}

$\equiv p \wedge (\neg q \vee r)$

↳ optionally distribute.

Example

Show $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$