Division rule

Recall: # of ways to eat 10 soups over 10 days = $10!$
($n!$ ways to put n things in order — permutations)

- # of ways to eat from 10 soups over 4 days $= 10\cdot 9\cdot 8\cdot 7$

$$= \frac{10!}{6!}$$

Q: how many ways to eat soup for 4 days if we don't care about order? ie how many subsets of 4 soups are there out of 10?

$10\cdot 9\cdot 8\cdot 7$ — too big. eg. it counts

chicken, tomato, mushroom, spinach

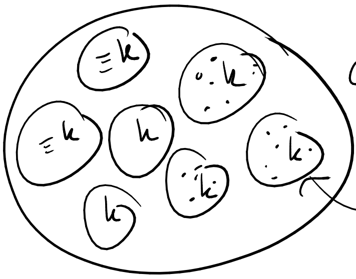
tomato, mushroom, chicken, spinach

as separate but we want to consider them the same.

Each set of 4 soups got counted $4! = 24$ times since that's the # of ways to put 4 things in some order.
So the answer we want is

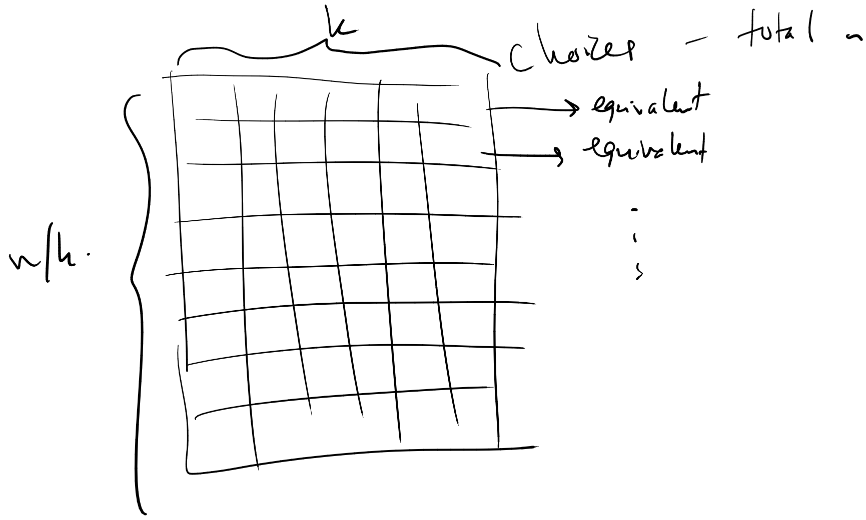
$$\frac{10\cdot 9\cdot 8\cdot 7}{4!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 210.$$

Division rule: Suppose we have n total choices but we want to think of some of them as being equivalent. If the choices always come in groups of exactly k equivalent choices, then the number of "really different" choices is n/k.



choices — n.

# of equivalence classes is n/k.



eg. How many ways are there to seat 5 people around a circular table if we only care who is sitting to the L/R of who?

If we do care specifically which seat each person is in, # of ways to seat them is $5!$.

For any ordering, there are 5 equivalent rotations.

So by the division rule, there are $\frac{5!}{5} = 4!$ ways to seat them.

eg. Consider pairs of numbers from the set $\{1, \dots, n\} \times \{1, \dots, n\}$.

$= \{(1,1), (1,2), \dots, (2,1), \dots (n,n)\}$.

There are n^2 of these.

How many are there if we don't care about the order? i.e. we would consider $(3,4)$ and $(4,3)$ the same.

Can't use division rule directly — pairs like $(3,4)$ and $(4,3)$ are counted twice, but eg. $(1,1)$, $(2,2)$, etc are only counted once.

There are $n^2 - n$ pairs (a,b) where $a \neq b$. Each of these is counted twice, so there are $\frac{n^2-n}{2}$ unordered pairs where $a \neq b$; we can then add back in the pairs like $(1,1), (2,2) \dots$

for a total of $n + \frac{n^2-n}{2} = \frac{2n}{2} + \frac{n^2-n}{2} = \frac{n^2+n}{2}$.

$= \frac{n(n+1)}{2}$.

eg. Back to Soup.

In general, how many ways would there be to pick k soups out of n if we don't care about the order?

Put another way, how many size-k subsets are there of a size-n set?

- $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$ ways to choose a <u>sequence</u> of k things

$$= \frac{n!}{(n-k)!}$$

- But this counts every ordering of the k things separately, so to get the # of <u>sets</u> we divide by $k!$

Hence $= \boxed{\dfrac{n!}{(n-k)!\,k!}}$.

"binomial coefficient", special notation:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

= # of size-k subsets of n things

= # of ways to choose k out of n things, if we don't care about order.

"n choose k"

eg. $$\binom{11}{3} = \frac{11!}{3!\,8!} = \frac{11 \cdot \overset{5}{\cancel{10}} \cdot \overset{3}{\cancel{9}}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 165$$

$$\binom{37}{4} = \frac{37!}{4!\,33!} = \frac{37 \cdot \overset{\cancel{9}\,3}{\cancel{36}} \cdot 35 \cdot \overset{17}{\cancel{34}}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 37 \cdot 35 \cdot 17 \cdot 3$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

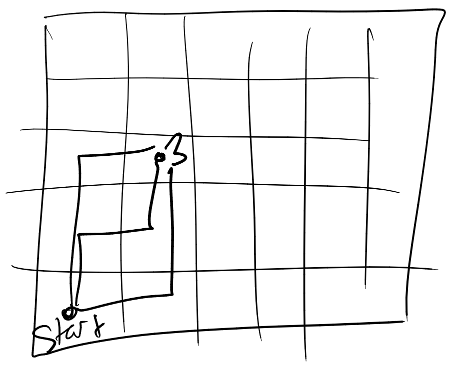
Pascal's Triangle

	n \ k	0	1	2	3	4	5	6	7	8
	0	1	0	0						
1	1	1	1	0						
1	2	1	2	1						
	3	1	3	3	1					
2	4	1	4	6	4	1				
3	5	1	5	10	10	5	1			
5	6	1	6	15	20	15	6	1		
8	7	1	7	21	35	35	21	7	1	
13	8	1	8	28	56	70	56	28	8	1

$$\binom{3}{0} = \frac{3!}{0!\,3!} = 1.$$

$$\binom{3}{1} = \frac{3!}{1!\cdot 2!} = 3.$$

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{{}^{3}\cancel{6}\cdot 5}{\cancel{2}\cdot 1} = 15$$



- symmetric $\leftrightarrow$ ✓
- all 1's on $\binom{n}{0}$ or $\binom{n}{n}$ $\leftarrow$ ✓
- Counting #'s on column for 1 $\leftarrow$ ✓
- Δ #'s for $\binom{\ }{2}$ $\leftarrow$
- add 2 to get # below?

①
$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\frac{n!}{0!\,n!} = \frac{n!}{n!\,0!} = 1.$$

1 way to choose all or nothing.

② Symmetric?
$$\binom{n}{k} = \binom{n}{n-k}$$

\# ways to choose k = # ways to choose n−k things we don't want

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!}$$
$$= \frac{n!}{(n-k)!\,k!}$$

(3) $\binom{n}{1} = \binom{n}{n-1} = n.$

n ways to choose 1 thing.

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cdot\cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

(4) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$?

(5) $\binom{n}{2} = \Delta_{n-1}$ ($n-1^{st}$ triangular #) $= 1+2+\dots+(n-1)$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} = \Delta_{n-1}.$$