Induction.

eg. $\forall n : \mathbb{N}.\ 1+2+3+\dots+n = \dfrac{n(n+1)}{2}$.

In general, to prove $\forall n : \mathbb{N}.\ P(n)$:

$$\underset{1}{P(0)} \to P(1) \to P(2) \xrightarrow[2]{} P(3) \to P(4) \to \dots$$

1. Prove $P(0)$ is true
2. Prove that each $P(k) \to P(k+1)$.

Formally:

$$\Big(\underset{1}{P(0)} \wedge \underset{2}{\big(\forall k : \mathbb{N}.\ P(k) \to P(k+1)\big)} \Big) \to \big(\forall n : \mathbb{N}.\ P(n)\big)$$

Principle of induction

eg. Prove $1+2+3+\dots+n = \dfrac{n(n+1)}{2}$ for all $n \geq 0$.

$P(n) =$ " $1+2+\dots+n = \dfrac{n(n+1)}{2}$ ". We want to prove $\forall n : \mathbb{N}.\ P(n)$, so we can use induction.

Hence we must prove $P(0) \wedge (\forall k : \mathbb{N}.\ P(k) \to P(k+1))$, so we will prove both.

induction hypothesis

(Base case). $P(0)$ says " $\underbrace{1+2+\dots+0}_{\text{add first 0 whole numbers}} = \dfrac{0(0+1)}{2}$ ".

This is true since both sides are 0.

Now we must prove $\forall k : \mathbb{N}.\ P(k) \to P(k+1)$. So let k be an arbitrary natural number; we must show $P(k) \to P(k+1)$.

So suppose $P(k)$ is true, that is, $\boxed{1+2+\dots+k = \dfrac{k(k+1)}{2}}$ KNOW

Then we will show $P(k+1)$, that is,

$\boxed{1+2+\dots+(k+1) = \dfrac{(k+1)(k+2)}{2}}$. WANT.

$1 + 2 + \dots + (k+1)$

$=$ {rewriting to emphasize k}.

$1 + 2 + \dots + k + (k+1)$

$=$ {assumption (induction hypothesis, IH)}.

$\frac{k(k+1)}{2} + (k+1)$

$=$ {algebra}

$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$=$ {algebra}

$\frac{k(k+1) + 2(k+1)}{2}$

$=$ {algebra}

$\frac{(k+1)(k+2)}{2}$

Therefore $P(k) \rightarrow P(k+1)$.

Since that was for an arbitrary $k \in \mathbb{N}$,
$\forall k : \mathbb{N}.\ P(k) \rightarrow P(k+1)$.

So, by induction, $\forall n : \mathbb{N}.\ P(n)$.

eg. Prove $2^n < n!$.

Let $Q(n) =$ "$2^n < n!$". Let's try to prove $\forall n : \mathbb{N}.\ Q(n)$ by induction.

X false.

$Q(0) : 2^0 < 0! \longrightarrow 1 < 1$ X

$Q(1) : 2^1 < 1! \longrightarrow 2 < 1$ X

$Q(2) : 2^2 < 2! \longrightarrow 4 < 2$ X

$Q(3) : 2^3 < 3! \longrightarrow 8 < 6$ X

$Q(4)$: $2^4 < 4! \longrightarrow 16 < 24$ ✓

Conjecture: $\forall n : \mathbb{N}. \ (n \geq 4) \longrightarrow Q(n)$.

$\forall n \geq 4. \ Q(n)$.

We can start induction at 4 instead of 0.

Proof

Base case: $Q(4)$ says $2^4 < 4! \equiv 16 < 24$ ✓.

Now we must show that for any $\boxed{k \geq 4,}$ KNOW. $Q(k) \rightarrow Q(k+1)$. So let k be an arbitrary natural number ≥ 4, and suppose $Q(k)$, that is, $\boxed{2^k < k! \ .}$ KNOW.

WANT.

We must show $Q(k+1)$, that is, $\boxed{2^{k+1} < (k+1)!}$

$$
\begin{aligned}
& 2^{k+1} \\
= \quad & \{\text{algebra}\} \\
& 2 \cdot 2^k \\
< \quad & \{\text{induction hypothesis} — 2^k < k!\}. \\
& 2 \cdot k! \\
< \quad & \{2 < k+1 \text{ because } k \geq 4\}. \\
& (k+1) \cdot k! \\
= \quad & \\
& (k+1)!
\end{aligned}
$$

So by transitivity of $<$, $2^{k+1} < (k+1)!$

Therefore, by induction, $2^n < n!$ for all $n \geq 4$.

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eg. Recall Fibonacci numbers:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Theorem. $F_n \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Proof (attempt): by induction.

Let $R(n) =$ "$F_n \leq 2^{n-1}$".

Base case: $R(0) =$ "$F_0 \leq 2^{0-1}$"

$\equiv 0 \leq \frac{1}{2}$ ✓.

Induction step: show $\forall k : \mathbb{N}.\ R(k) \rightarrow R(k+1)$.

So let k be an arbitrary natural number, and suppose $R(k)$, that is, $\boxed{F_k \leq 2^{k-1}.}$ KNOW.

We must show $R(k+1)$, that is, $\boxed{F_{k+1} \leq 2^k.}$ WANT.

F_{k+1}

$=$ { defn of Fib. #'s } ✗

$F_k + F_{k-1}$

Not allowed — k could be 0, and $F_1 = F_0 + F_{-1}$ is not valid.

$\le$ {induction hypothesis — $F_k \le 2^{k-1}$}.

$2^{k-1} + F_{k-1}$

stuck — don't know anything about F_{k-1}.

<u>Strong induction</u>.

$$P(0) \to P(1) \to P(2) \quad P(3) \to P(4) \quad \dots$$

Idea: to show each $P(k)$, use not just the one previous, but <u>all</u> previous ones.

Formally:

$$\Big(\underline{\forall k : \mathbb{N}.\ (\forall j < k.\ P(j)) \to P(k)} \Big) \longrightarrow \Big(\forall n : \mathbb{N}.\ P(n) \Big)$$

So when doing a proof by induction, we are allowed to assume <u>all</u> previous $P(j)$ for $j < k$ when proving $P(k)$.

Theorem $F_n \le 2^{n-1}$ for all $n \in \mathbb{N}$.

Proof by strong induction. We also need 2 base cases since each inductive step needs the 2 previous Fibonacci numbers.

$R(0) \equiv$ "$F_0 \le 2^{0-1}$" ✓

$R(1) \equiv$ "$F_1 \le 2^{1-1}$" $\equiv 1 \le 1$ ✓

Now let k be an arbitrary natural number with $k \geq 2$, and suppose $R(j)$ is true for <u>all</u> $j < k$.

We must show $R(k)$, that is, $\underline{\underline{F_k \le 2^{k-1}}}$.

$$\begin{aligned}
& F_k \\
= \quad & \{\text{def'n of } F_k, \text{ since } k \geq 2\} \\
& F_{k-1} + F_{k-2} \\
\le \quad & \{\text{assumed } R(k-1) \text{ and } R(k-2)\} \\
& 2^{k-2} + \underline{2^{k-3}} \\
\le \quad & \{2^{k-3} \le 2^{k-2}\} \\
& 2^{k-2} + 2^{k-2} \\
= \quad & 2 \cdot 2^{k-2} = 2^{k-1}.
\end{aligned}$$

Challenge: $(3/2)^n < F_n$ for suitable n.

Ex. Let $a_0 = 0$, $a_n = 2a_{n-1} + 1$.

$0, 1, 3, 7, 15, \ldots$ $P(n)$.

Prove: $\boxed{a_n = 2^n - 1}$ for all $n \geq 0$.

By induction.

- Base case: $P(0)$ says $a_0 = 2^0 - 1$. ✓

$$a_0 = 0, \quad 2^0 - 1 = 1 - 1 = 0.$$

- Now we must show that $P(k) \to P(k+1)$ for all $k \geq 0$. So let k be arbitrary natural number, and suppose $P(k)$, that is, $\boxed{a_k = 2^k - 1.}$ KNOW.

WANT.

We must show $P(k+1)$, ie. $\boxed{a_{k+1} = 2^{k+1} - 1.}$

$$
\begin{aligned}
& a_{k+1} \\
= & \quad \{\text{by definition}\} \\
& 2a_k + 1 \\
= & \quad \{\text{assumption (induction hypothesis)}\} \\
& 2(2^k - 1) + 1 \\
= & \ 2^{k+1} - 2 + 1 = 2^{k+1} - 1 \quad ✓
\end{aligned}
$$

Ex. Prove: the sum of the first n odd natural numbers is n^2.

eg $1+3+5 = 3^2$

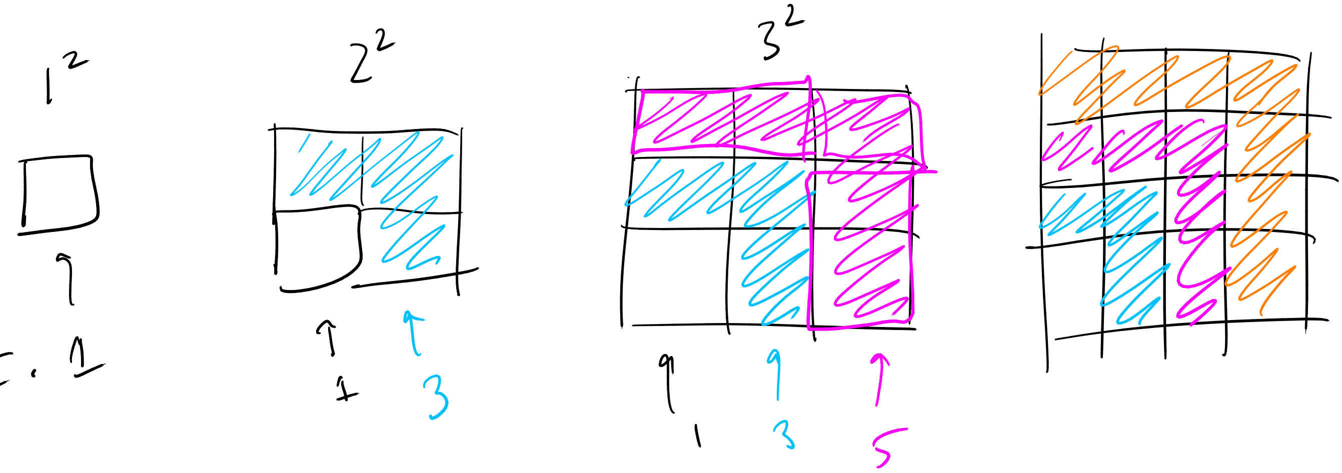
Let $P(n)$ = "$1+3+\dots+(2n-1) = n^2$".
We will prove $\forall n:\mathbb{N}.\ P(n)$ by induction.

Proof.

- Base case: $P(0)$ just says $0 = 0^2$. ✓
- Induction step: we must show $\forall k:\mathbb{N}.\ P(k) \to P(k+1)$.
  Let k be arbitrary and suppose $P(k)$, ie.
  $\boxed{1+3+\dots+(2k-1) = k^2}$ KNOW. We must show
  $P(k+1)$, ie. $\boxed{1+3+\dots+(2(k+1)-1) = (k+1)^2}$. WANT.

$$
\begin{aligned}
& 1+3+\dots+(2(k+1)-1) \\
= \;& \{\text{algebra}\} \\
& 1+3+\dots+(2k+1) \\
= \;& \{\text{assumption}\}. \\
& k^2 + (2k+1) \\
= \;& \{\text{algebra}\}. \\
& (k+1)^2 \quad ✓.
\end{aligned}
$$

Proof 2.



Proof 3.

$$\sum_{1\le k\le n}(2k-1) = 2\left(\sum_{1\le k\le n} k\right) - \left(\sum_{1\le k\le n} 1\right)$$

$$= \cancel{2}\left(\frac{n(n+1)}{\cancel{2}}\right) - n$$

$$= n(n+1) - n$$

$$= n^2 + n - n = n^2.$$