Solving recurrences

How to turn a recurrence → closed form?

One technique: unfold the recurrence + look for patterns.

e.g. $g(0) = 1$

$g(n) = g(n-1) + 2 \quad (n \geq 1)$

$1, 3, 5, 7, 9, 11, \ldots$

$$g(n) = g(n-1) + 2 = (g(n-2) + 2) + 2$$

$$= g(n-3) + 2 + 2 + 2$$

$$= g(n-4) + 2 + 2 + 2 + 2$$

$$\ldots$$

$$= g(n-k) + \underbrace{2 + 2 + \ldots + 2}_{k \text{ 2's.}}$$

algebra

thinking!

general pattern

$$= g(n-k) + 2k.$$

substitute n for k (because we know $g(0)$)

$$= g(n-n) + 2n$$

$$= g(0) + 2n$$

$$= 1 + 2n.$$

algebra

eg. $\underline{p(0) = 0}$, $\underline{p(n) = 2p(n-1) + 1}$

$$
\begin{aligned}
p(n) &= 2p(n-1) + 1 \\
&= 2(2p(n-2) + 1) + 1 \\
&= 4p(n-2) + 2 + 1 \\
&= 4(2p(n-3) + 1) + 2 + 1 \\
&= 8p(n-3) + 4 + 2 + 1 \\
&= 16p(n-4) + 8 + 4 + 2 + 1 \\
\ldots &= 2^k p(n-k) + \boxed{2^{k-1} + 2^{k-2} + \ldots + 2 + 1} \\
&= 2^k p(n-k) + 2^k - 1 \\
&= 2^n p(0) + 2^n - 1 \qquad \text{(subst. } k=n\text{).} \\
&= 2^n - 1.
\end{aligned}
$$

Sum of a geometric sequence with $r=2$

$$\frac{1-r^{n+1}}{1-r} = \frac{1-2^k}{1-2} = 2^k - 1$$

eg. $q(0) = 1, \quad q(n) = 3q(n-1) + 2.$

$1, 5, 17, 53, 161, \ldots$?

$$
\begin{aligned}
q(n) &= 3q(n-1) + 2 \\
&= 3\left(3q(n-2) + 2\right) + 2 \\
&= 3^2 q(n-2) + 3\cdot 2 + 2 \\
&= 3^2\left(3q(n-3) + 2\right) + 3\cdot 2 + 2 \\
&= 3^3 q(n-3) + 3^2\cdot 2 + 3\cdot 2 + 2 \\
&= 3^4 q(n-4) + 3^3\cdot 2 + 3^2\cdot 2 + 3\cdot 2 + 2. \\
&= 3^k q(n-k) + 3^{k-1}\cdot 2 + 3^{k-2}\cdot 2 + \cdots + 2. \\
&= 3^k q(n-k) + 2\underbrace{\left(3^{k-1} + 3^{k-2} + \cdots + 1\right)}_{\frac{1-3^k}{1-3} = \frac{3^k - 1}{2}} \\
&= 3^k q(n-k) + 3^k - 1
\end{aligned}
$$

$$= 3^n q(0) + 3^n - 1$$
$$= 3^n + 3^n - 1$$
$$= \underline{2 \cdot 3^n - 1}$$

To check this is correct, check that this formula makes the recurrence true.

eg. Triangular numbers:

$$t(0) = 0$$
$$t(n) = t(n-1) + n.$$

Try it!

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}. \quad (n \geq 2).$$

$$\hookrightarrow \boxed{F_n - F_{n-1} - F_{n-2}} = 0.$$

Consider $x^2 - x - 1 = 0.$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Call $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$

Notice $\varphi^2 - \varphi - 1 = 0$, and therefore

$$\varphi^n - \varphi^{n-1} - \varphi^{n-2} = 0.$$

(same for ψ).

So $f(n) = \varphi^n$ satisfies same recurrence as Fibonacci #'s, i.e.

$$f(n) = f(n-1) + f(n-2).$$

But $f(0) = \varphi^0 = 1 \neq F_0$

$g(n) = \psi^n$ also satisfies the recurrence

Claim: $af(n) + bg(n)$ satisfies the recurrence for any a, b.

Why?

$$\left(a\, f(n-1) + b\, g(n-1)\right) + \left(a\, f(n-2) + b\, g(n-2)\right)$$

$$= a\left(f(n-1) + f(n-2)\right) + b\left(g(n-1) + g(n-2)\right)$$

$$= \underline{a f(n) + b g(n)}.$$

Now find a, b that give us Fibonacci #'s

$$a \cdot \varphi^0 + b\, \psi^0 = F_0 = 0$$

$$a \cdot \varphi^1 + b\, \psi^1 = F_1 = 1.$$

$$\Rightarrow \quad a + b = 0.$$

$$a \cdot \varphi + b \cdot \psi = 1$$

$$\hookrightarrow a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solve system of 2 equations for a, b.

Get $a = \frac{1}{\sqrt{5}}$ $\quad b = -\frac{1}{\sqrt{5}}$.

Hence $F_n = \dfrac{\varphi^n}{\sqrt{5}} - \dfrac{\psi^n}{\sqrt{5}}$.