Sequences

A sequence is (informally) a list of things (typically numbers),

e.g. $\underline{2, 5, 8, 11, 14,} \ldots$

Defn A **sequence** is a function from $\mathbb{N}$ (or a subset of $\mathbb{N}$) to some set A. We use subscript notation (like a_n, b_n) to denote the image of $n \in \mathbb{N}$, e.g. a_n is the n^{th} item in the sequence.

(Some people use notation $\{a_n\}$ but this is terrible notation.)

eg. Consider the sequence of rational numbers with $a_n = 1/n$, starting at a_1. This sequence starts

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

eg. The sequence defined by $a_n = 5$ begins:

$$5, 5, 5, 5, 5, \ldots$$

eg. The string "hello" can be thought of as a finite sequence, with

$$s_0 = h, \quad s_1 = e, \quad s_2 = l, \quad s_3 = l, \quad s_4 = o.$$

eg. Consider the sequence

$$2, 5, 8, 11, 14, 17, \ldots$$

Each term is 3 more than the previous term.

(arithmetic sequence = difference between terms = constant)

We could define this by

$$a_n = 3n - 1, \text{ starting with } a_1$$

OR $$a_n = 3n + 2, \text{ starting with } a_0.$$

"closed form" = direct formula for each term.

We could also define this sequence using a <u>recurrence relation</u> (aka <u>recurrence</u>):

$$a_0 = 2$$
$$a_n = a_{n-1} + 3 \qquad (n \geq 1)$$

recurrence

eg. Consider the sequence

$$2, 6, 18, 54, 162, \ldots$$

Each term is 3x the previous term. (geometric sequence)

We can define it by

$$a_0 = 2$$
$$a_n = a_{n-1} \times 3 \qquad (n \geq 1).$$

or, in closed form,

$$a_n = 2 \cdot 3^n.$$

eg. Let $a_0, a_1, \ldots$ be the sequence defined by

$$a_0 = 0$$
$$\underline{a_n = 2a_{n-1} + 1} \qquad (n \geq 1).$$

We can calculate:

$$a_0 = 0$$
$$a_1 = 2a_0 + 1 = 1$$
$$a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 3.$$
$$a_3 = 7$$
$$a_4 = 15$$
$$a_5 = 31$$
$$a_6 = 63$$
$$\vdots$$

Conjecture: $\underline{a_n = 2^n - 1} = 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 2^0.$

We can verify that this closed form is correct by checking that it satisfies the recurrence.

- $a_0 = 2^0 - 1 = 1 - 1 = 0$ ✓
- Check: does $a_n = 2a_{n-1} + 1$ when we substitute our formula?

$$2^n - 1 \stackrel{?}{=} 2(2^{n-1} - 1) + 1$$
$$= 2 \cdot 2^{n-1} - 2 + 1$$
$$= 2^n - 1$$

✓

eg. $a_0 = 1, \quad a_n = n \cdot a_{n-1} \quad (n \geq 1).$

$$a_0 = 1$$
$$a_1 = 1$$
$$a_2 = 2$$
$$a_3 = 3 \cdot 2 = 6$$
$$a_4 = 4 \cdot 6 = 24$$
$$a_5 = 5 \cdot 24 = 120.$$

In fact $a_n = n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

eg. Describe the following sequences with either a recurrence or a closed form:

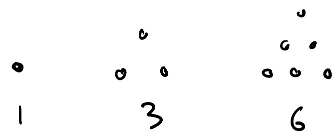
ⓐ $1, 3, 5, 7, 9, 11, 13, \ldots$ Odds

$a_0 = 1$
$a_n = a_{n-1} + 2 \quad (n \geq 1)$

$a_n = 2n + 1$ starting at a_0

ⓑ $0, 1, 3, 6, 10, 15, 21, \ldots$ Triangular #'s.

$a_0 = 0$
$a_n = a_{n-1} + n$



$\left(a_n = \dfrac{n(n+1)}{2} \right)$

ⓒ $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ Fibonacci #'s.

$a_0 = 0$
$a_1 = 1$
$a_n = a_{n-1} + a_{n-2}$