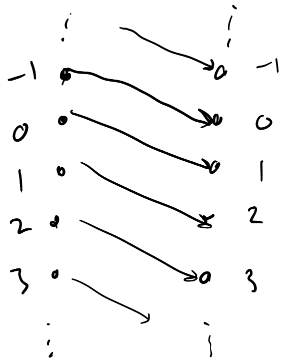
Def'n A function $f: A \to B$ is <u>invertible</u> (a <u>bijection</u>) if it is both one-to-one and onto. We write f^{-1} to represent the inverse function of f, that is, $f^{-1}: B \to A$ such that

$$\big(f(a) = b\big) \longleftrightarrow \big(f^{-1}(b) = a\big).$$

(Caution: f^{-1} does <u>not</u> mean $\frac{1}{f}$)

Remark. To prove a function is a bijection, we can show it is both one-to-one + onto. Alternatively, we can show that there is a function g such that $f(g(b)) = b$ for all b <u>and</u> $g(f(a)) = a$ for all a.

(Exercise: prove this works.)



Ex.

Which are bijections?

$f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$ ✓

$g: \mathbb{N} \to \mathbb{N}$, $f(n) = \lfloor n/2 \rfloor$ ✗ not 1-1

$h: \mathbb{Z} \to \mathbb{Z}$, $h(x) = 3x + 2$ ✗ not onto.

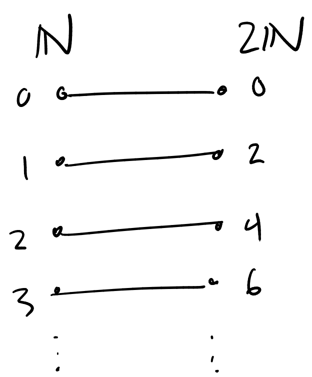
$q: \mathbb{Q} \to \mathbb{Q}$, $q(x) = 3x + 2$ ✓

$$q^{-1}(y) = \frac{y-2}{3}.$$

$f: \mathbb{N} \to \mathbb{N}$
$f(n) = 2n$ — not onto

$2\mathbb{N} = \{0, 2, 4, 6, \ldots\}$

$f: \mathbb{N} \to 2\mathbb{N}$
$f(n) = 2n$ — is a bijection.



$\mathbb{N}$ and $2\mathbb{N}$ "have the same size" since we can match them up!?

Def'n Let A, B be sets. If there exists a bijection $f: A \to B$ then we say A and B have the same cardinality, and write

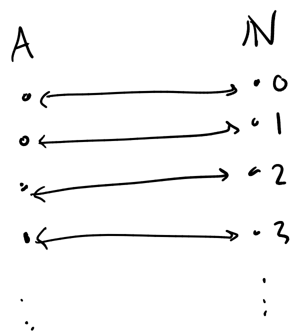
$$|A| = |B|.$$

(Challenge: show this is an equivalence relation)

Likewise, if there is an injection $f: A \to B$, we say

$$|A| \leq |B|.$$

Def'n A set A is countable if it is either finite or has the same cardinality as $\mathbb{N}$, (ie. there is a bijection $A \to \mathbb{N}$).



ie. put elements of A in a list, or program to print all elements of A eventually.

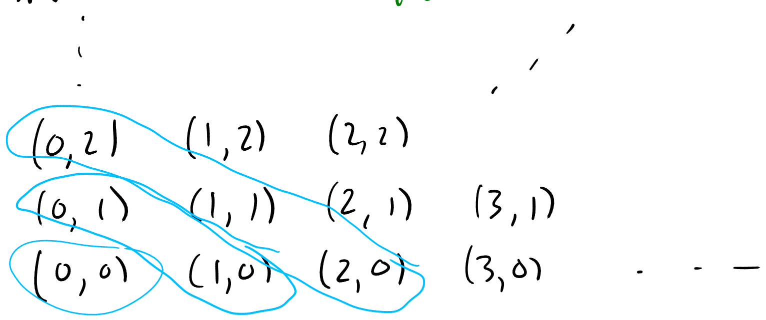
eg. The set of even natural #s $2\mathbb{N} = \{0, 2, 4, 6, \dots\}$ is countable.
$f: \mathbb{N} \to 2\mathbb{N}$, $f(n) = 2n$ is a bijection.

eg. Is $\mathbb{Z}$ countable? Yes!

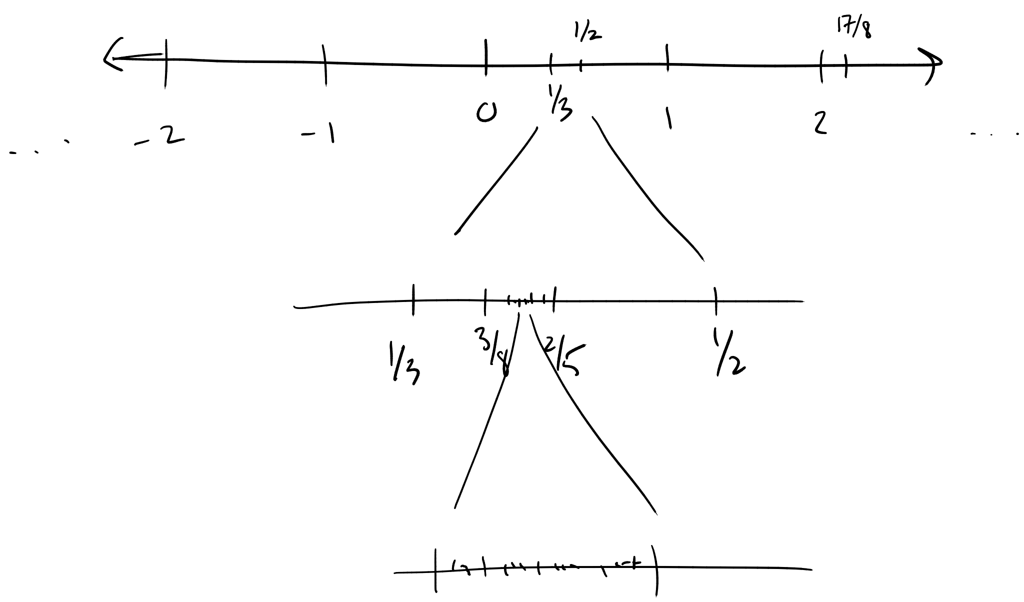
$$\begin{array}{ccc} \mathbb{Z} & & \mathbb{N} \\ 0 & \leftrightarrow & 0 \\ 1 & \leftrightarrow & 1 \\ -1 & \leftrightarrow & 2 \\ 2 & \leftrightarrow & 3 \\ -2 & \leftrightarrow & 4 \\ 3 & \leftrightarrow & 5 \\ -3 & \leftrightarrow & 6 \\ \vdots & & \end{array}$$

this is a bijection $\mathbb{Z} \leftrightarrow \mathbb{N}$.

eg. Is $\mathbb{N} \times \mathbb{N}$ countable? Yes!

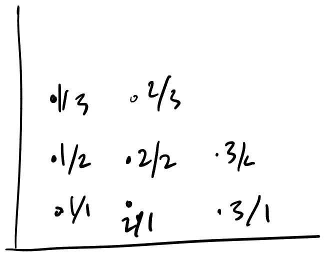


$$\begin{array}{ccc} \mathbb{N} \times \mathbb{N} & & \mathbb{N} \\ (0,0) & \leftrightarrow & 0 \\ (1,0) & \leftrightarrow & 1 \\ (0,1) & \leftrightarrow & 2 \\ (2,0) & \leftrightarrow & 3 \\ (1,1) & \leftrightarrow & 4 \\ (0,2) & \leftrightarrow & 5 \\ \vdots & & \end{array}$$



Is $\mathbb{Q}$ countable? Yes!

rational # is like a pair of integers.



$$|\mathbb{Q}| \le |\mathbb{Z}\times\mathbb{Z}| = |\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$$

It is possible to write a program to list all rational numbers!

An uncountable set! $\mathbb{R}$.

Suppose we have *any* function $f: \mathbb{N} \to \mathbb{R}$. We will show it is not onto. i.e. Suppose we have a list of real #s, show some real # is not on the list. Actually do this for just $[0,1)$.

0.1234582791 . . . .
0.314159265 3 . . .
0.1000000000 . . . .
0.33333 3333 3 . . .
0.888888 . . . .
⋮

Look @ 1st digit of 1st #,
2nd digit of 2nd #,
⋮
kth digit of kth #

Pick a number whose kth digit is different than the kth digit of kth #.

e.g. 0.888828 . . .

This # is different than each of the #s on the list.

Hence, $\mathbb{R}$ is uncountable!
There are more real #s than natural numbers.