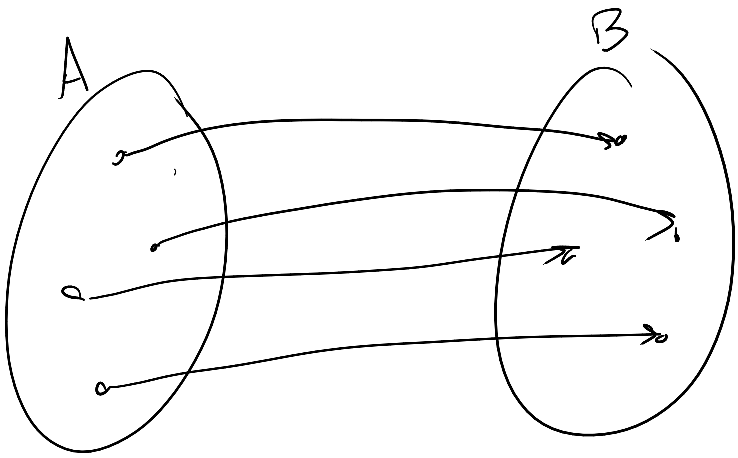
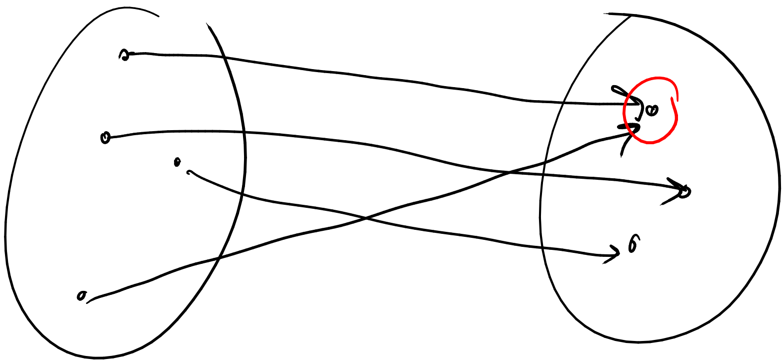
Motivation: Inverting / undoing functions.

Can't do this w/ every function! When can we "invert" a function and get another function?



✓

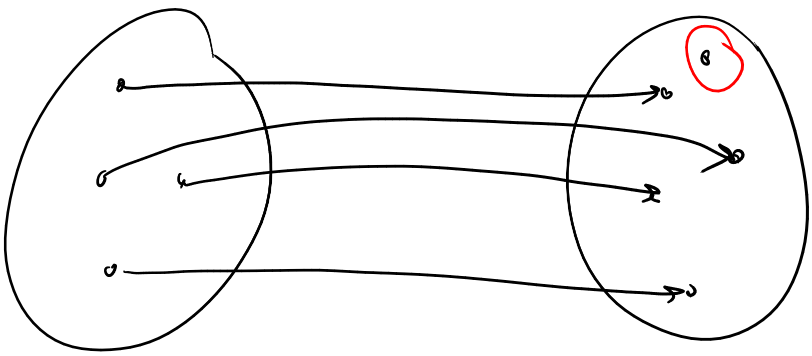
(A)



✗

Can't invert — would need 2 outputs for 1 input

(B)



✗

Can't invert — would end up w/ undefined on one input.

Def'n A function $f : A \to B$ is one-to-one (injective) iff no two different elements of the domain map to the same element of the codomain. Formally, A

$$\forall a_1, a_2 : A.\ a_1 \neq a_2 \longrightarrow f(a_1) \neq f(a_2).$$

Or, equivalently,

$$\forall a_1, a_2 : A.\ f(a_1) = f(a_2) \longrightarrow a_1 = a_2.$$

We typically use this second form to prove a given function 1-1.

How would we show a given function is not 1-1?

$$\neg \left(\forall a_1, a_2 : A.\ f(a_1) = f(a_2) \longrightarrow a_1 = a_2 \right)$$

$$\equiv \exists a_1, a_2 : A.\ \neg \left(f(a_1) = f(a_2) \longrightarrow a_1 = a_2 \right)$$

$$\equiv \exists a_1, a_2 : A.\ \neg \left(f(a_1) \neq f(a_2) \lor a_1 = a_2 \right)$$

$$\equiv \exists a_1, a_2 : A.\ f(a_1) = f(a_2) \land a_1 \neq a_2.$$

i.e. find 2 distinct elts. of A which map to the same output.

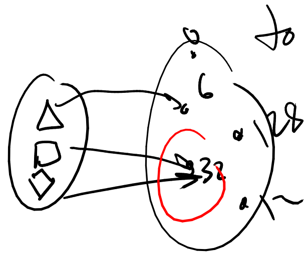
eg. $g : Shp \to \mathbb{N}$

$g(\triangle) = 6$

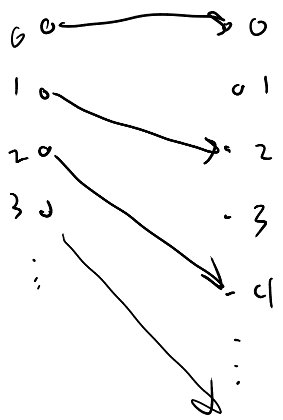
$g(\square) = 32$

$g(\diamond) = 32$

Not 1-1 since $\square, \diamond$ map to same output.



eg. $h : \mathbb{N} \to \mathbb{N}$
$h(n) = 2n.$



h is 1-1.

Proof. To show:

$$\forall a_1, a_2 : \mathbb{N}.\ h(a_1) = h(a_2) \to a_1 = a_2.$$

Let a_1, a_2 be arbitrary natural numbers, and suppose $h(a_1) = h(a_2)$. We will show $a_1 = a_2$.

> $h(a_1) = h(a_2)$ means $2a_1 = 2a_2$.
> Dividing both sides by 2 yields $a_1 = a_2$.

eg $l : \mathbb{N} \to \text{Shp}$

$l(3n) = \diamond$

$l(3n+1) = \square$

$l(3n+2) = \triangle$

eg $l(6) = l(9)$ so, not 1-1.

Def'n A function $f : A \to B$ is <u>onto</u> (surjective) if "every element of B is covered/hit", ie.

B

$$\forall b : B.\ \exists a : A.\ f(a) = b.$$

ie. range is the entire codomain.

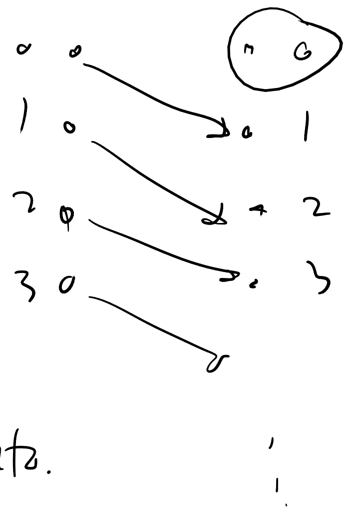
eg. $g: Shp \to \mathbb{N}$ from before is <u>not</u> onto; eg. $12 \in \mathbb{N}$ but it is never the output of g for any input.

eg. $h: \mathbb{N} \to \mathbb{N}$, $h(n) = 2n$ — <u>not</u> onto. No odd #'s are output.

eg. $l : \mathbb{N} \to Shp$ — yes, onto, every possible element of Shp is an output for some input.

$q : \mathbb{N} \to \mathbb{N}$

$q(n) = n + 1$ — not onto



$p : \mathbb{Z} \to \mathbb{Z}$

$p(n) = n + 1$. — onto.