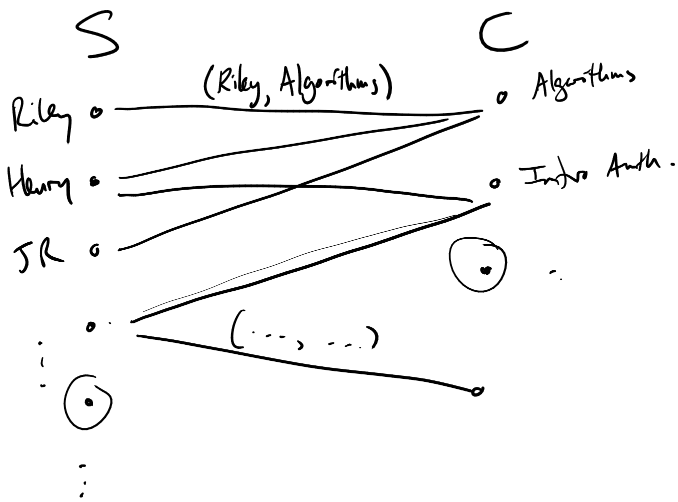
# Relations

e.g. Let $S$ be the set of students at Hendrix, and $C$ the set of classes offered next year. We want to model which students are registered for which classes.



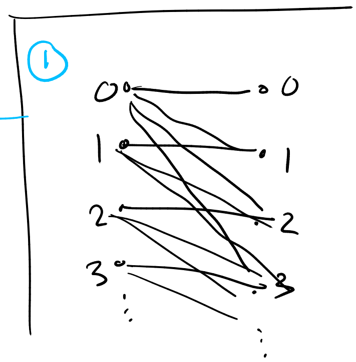

**Def'n** Let $A$ and $B$ be sets. A (binary) relation from $A$ to $B$ is a subset of $A \times B$. If $R$ is a relation, we write $a R b$ to mean $(a,b) \in R$.

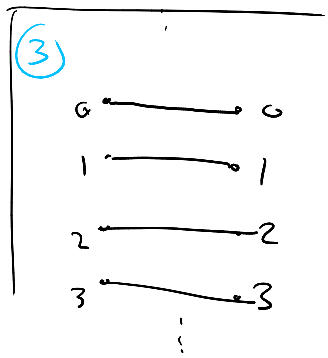
Often we care about relations from a set to itself.

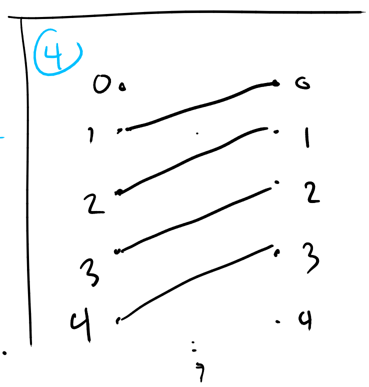
**Def'n** A relation on a set $A$ is a relation from $A$ to $A$ (ie a subset of $A \times A$).

e.g. Some relations on $\mathbb{N}$:

1. $\{(a,b) \mid a \le b\}$ "the $\le$ relation"
2. $\{(a,b) \mid a > b\}$ "the $>$ relation"
3. $\{(a,b) \mid a = b\}$
4. $\{(a,b) \mid a = b+1\}$
5. $\{(a,b) \mid a+b \le 10\}$
6. $\{(a,b) \mid a \text{ evenly divides } b\}$
7. $\{(a,b) \mid a, b \text{ have the same last digit in base } 10\}$



Other examples:

- (8) $\{(P,Q) \mid P, Q \text{ Proposition where } P \to Q \text{ is true}\}$
- (9) $\{(P,Q) \mid P \leftrightarrow Q \text{ is true}\}$
- (10) $\{(S,T) \mid S \subseteq T\}$.

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Some special properties relations can have:

<u>Defn</u> A relation $R$ on a set $A$ is <u>reflexive</u> if

$$\forall a : A.\ a\,R\,a.$$

e.g. $\leq$, $=$, $\subseteq$, $\to$, $\leftrightarrow$, evenly divides, same last digit

<u>Defn</u> A relation $R$ on a set $A$ is <u>symmetric</u> if

$$\forall a : A.\ \forall b : A.\ a\,R\,b \to b\,R\,a.$$

e.g. (3), (5), (7), (9)

<u>Defn</u> A relation $R$ on a set $A$ is <u>antisymmetric</u>

if

$$\forall a : A, \forall b : A.\ (a\,R\,b) \wedge (b\,R\,a) \longrightarrow (a = b).$$

<u>Defn</u> A relation $R$ on a set $A$ is <u>transitive</u>

if

$$\forall a,b,c : A.\ (a\,R\,b) \wedge (b\,R\,c) \longrightarrow (a\,R\,c).$$

**Def'n** An <u>equivalence relation</u> on a set $A$ is a reflexive, symmetric, transitive relation.

Equivalence relations are "equality-like" in some suitable sense.

eg. equality itself.

eg. same last digit.

eg. $\leftrightarrow$.

eg. Relation $L$ on English words, $w_1 \, L \, w_2$ if they have the number of letters. eg cow $L$ dog.

eg. Divisibility? No.

- reflexive? ✓
- transitive? ✓
- symmetric ✗

eg. $\{(a, b) \mid |a-b| \leq 1\}$.

- reflexive ✓
- symmetric ✓
- transitive ✗

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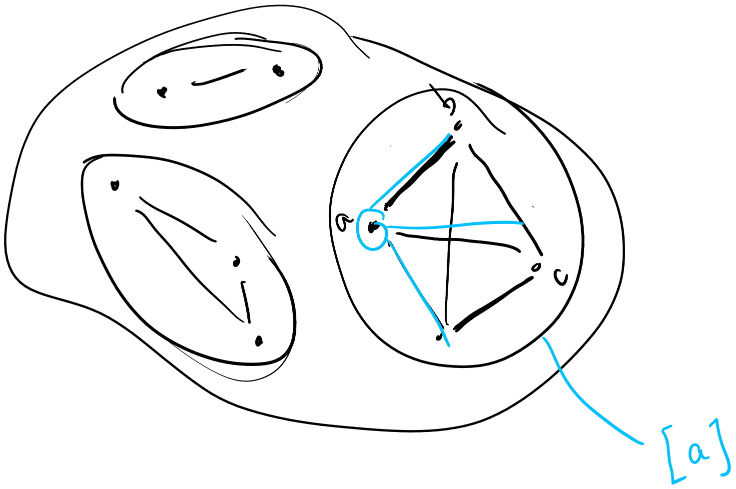


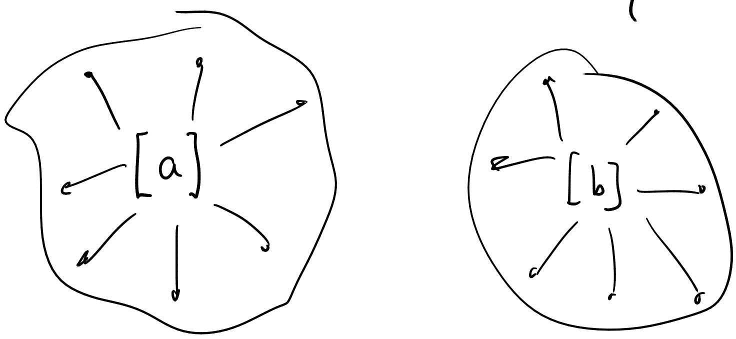
**Def'n** Let $\sim$ be an equivalence relation on $A$, and let $a \in A$. The <u>equivalence class</u> of $a$, written $[a]$, is the set of all elements of $A$ which are related to $a$, ie.

$$[a] = \{ b \mid b \in A,\ a \sim b \}.$$

eg. Under $L$, $[\text{dog}] = \{\text{dog, cat, cow, bat, pig, ant}, \dots\}$

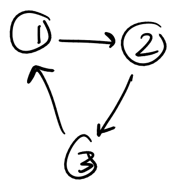
eg. Under "same last digit", $[13] = \{23, 33, 43, 3, 9723, \dots\}$

$$= \{10x + 3 \mid x \in \mathbb{N}\}.$$




**Theorem** Let $\sim$ be an equivalence relation on a set $A$. <u>For any $a, b \in A$</u>, the following three propositions are logically equivalent:

1. $a \sim b$
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$.



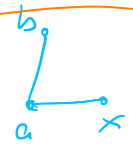

**Proof.** We will prove the chain of implications $1 \to 2 \to 3 \to 1$, which (by transitivity of $\to$) shows they are all equivalent.

Let $a, b$ be arbitrary elements of $A$.

$(1 \to 2)$. Suppose $\boxed{a \sim b}$; we must show $[a] = [b]$, which we can do by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

To show $[a] \subseteq [b]$, let $x \in [a]$, we must show $\boxed{x \in [b].}$ ✓

If $x \in [a]$, then by definition, $\boxed{a \sim x.}$
know



Since $\sim$ is symmetric, $b \sim a$. Since it is transitive, $b \sim a$ and $a \sim x$ means $b \sim x$, so by definition, $x \in [b]$.

To show $[b] \subseteq [a]$ uses exactly the same argument.

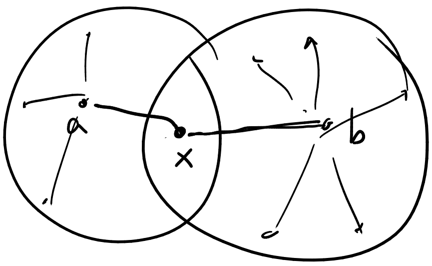
Therefore $[a] = [b]$ under the assumption $a \sim b$, so ① $\to$ ②.

$2 \to 3$. Suppose $[a] = [b]$; we must show they have something in common.

$$[a] \cap [b] = [a] \cap [a] = \underline{[a]}.$$

$[a] \neq \emptyset$ because $a \sim a$ by reflexivity.

$3 \to 1$ Proof by picture.




if $[a]$, $[b]$ overlap, there is some $x$ in both, so $a \sim x \sim b$ and $a \sim b$ by transitivity.