Def'n A set S is a subset of a set T if every element of S is also an element of T. That is,

$$\forall s:S.\ s \in T.$$

When S is a subset of T we write $S \subseteq T$.

eg. $\{1,3,6\} \subseteq \{2,3,7,6,1\}$.

$\{1,3,6\} \subseteq \mathbb{N}$.

$\{1,3,6\} \not\subseteq \{2,7,6,1\}$

Q. Is a set a subset of itself? ie is $S \subseteq S$? Yes!

Def'n $S = T$ when $(S \subseteq T)$ and $(T \subseteq S)$.

eg. Let $A = \{2x+3 \mid x \in \mathbb{Z}\}$ and $B = \{x \mid x \in \mathbb{Z},\ odd(x)\}$.
Prove $A = B$.

Proof: By definition, $A=B$ means $(A \subseteq B) \wedge (B \subseteq A)$.

First, we will show $A \subseteq B$, that is, $\forall a:A.\ a \in B$. So let s be an arbitrary element of A; we must show $s \in B$.

Since $s \in A$, it must be $s = 2x+3$ for some $x \in \mathbb{Z}$.
So s is an integer and it is odd since $2x+3 = 2(x+1)+1$.
Hence $s \in B$ by definition of B.

Show $B \subseteq A$ ...

Def'n The empty set, written $\emptyset$ or $\{\}$, is the set with no elements.

Q. Is $\emptyset \subseteq \mathbb{N}$? Yes! In fact $\emptyset \subseteq S$ for any set S.

By definition, this means

$$\forall a : \emptyset .\ a \in \mathbb{N}.$$

$$\forall a : \emptyset .\ a \in \mathbb{N} \equiv \forall a : D .\ (a \in \emptyset) \to (a \in \mathbb{N}).$$

True!