Defn. A **predicate** is a "proposition with inputs". (i.e. a function that outputs a proposition).

Quantifiers ($\forall$, $\exists$)

Def'n If $P(x)$ is a predicate and D is some "domain"/set/type of values, then

upside-down A for "All"

$$\forall x : D.\ P(x)$$

is a proposition, pronounced "for all x in D, $P(x)$ is true", which is true iff $P(x)$ is true for every possible x in the domain D.

[Sometimes written $\forall x \in D.\ P(x)$]

We should think of $\forall$ like a big AND.

$$\forall x:D.\ P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land \ldots$$

<u>Def'n</u> If $P(x)$ is a predicate and D a domain,

$$\exists x:D.\ P(x)$$

is a proposition, pronounced "there exists an x in D such that $P(x)$ is true"; which is true when there is at least one value of x in D which makes $P(x)$ true.

We can think of $\exists$ as a big OR:

$$\exists x:D.\ P(x) \equiv P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots$$

De Morgan laws for quantifiers

$$\neg(\forall x : D.\ \underline{P(x)}) \equiv \exists x : D.\ \neg P(x)$$

$$\begin{aligned}\neg(\forall x : D.\ P(x)) &\equiv \neg(P(x_1) \wedge P(x_2) \wedge \dots) \\ &\equiv \neg P(x_1) \vee \neg(P(x_2) \wedge \dots) \\ &\equiv \neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \\ &\qquad \vee \dots \\ &\equiv \exists x : D.\ \neg P(x).\end{aligned}$$

$$\neg(\exists x : D.\ P(x)) \equiv \forall x : D.\ \neg P(x)$$

Ex. Simplify:

$$\neg\left(\forall a : \mathbb{N}.\ \underline{\left(\exists b : \mathbb{N}.\ (a=0) \vee (a+b=0)\right)}\right)$$

$$\equiv \exists a : \mathbb{N}.\ \neg\left(\exists b : \mathbb{N}.\ (a=0) \vee (a+b=0)\right)$$

$$\equiv \exists a:\mathbb{N}.\ \forall b:\mathbb{N}.\ \neg\big((a=0) \vee (a+b=0)\big)$$

$$\equiv \exists a:\mathbb{N}.\ \forall b:\mathbb{N}.\ \neg(a=0) \wedge \neg(a+b=0)$$

$$\equiv \exists a:\mathbb{N}.\ \forall b:\mathbb{N}.\ a\neq 0 \wedge a+b\neq 0.$$

Ex: translate English into prop. logic.

① "$\boxed{\text{Every}}$ natural number is less than or equal to its own square."

$$\forall x:\mathbb{N}.\ x \leq x^2$$

② "The square of $\boxed{\text{any}}$ integer is nonnegative."

$$\forall x:\mathbb{Z}.\ x^2 \geq 0$$

"some integer..."

$$\exists x:\mathbb{Z}.\ x^2 \geq 0$$

③ "1369 is a perfect square."

$$\exists x:\mathbb{N}\ .\ x^2 = 1369.$$

④ "n is even."

$$\underline{Even(n)} = \exists k : \mathbb{Z}.\ n = 2k.$$

⑤ "n is odd."

$$Odd(n) = \exists k : \mathbb{Z}.\ n = 2k + 1$$

OR

$$odd(n) = \neg(\exists k : \mathbb{Z}.\ n = 2k)$$

$$\equiv \forall k : \mathbb{Z}.\ n \neq 2k$$

$$\equiv \neg(Even(n)).$$

⑥ "If n is even, then n+2 is even."

$$\forall n : \mathbb{Z}.\ Even(n) \rightarrow Even(n+2)$$

⑦ "Every even integer equals 2."

$$\forall n : (\text{even integers}).\ n = 2$$

simple, direct, but unsatisfactory – hides too much.

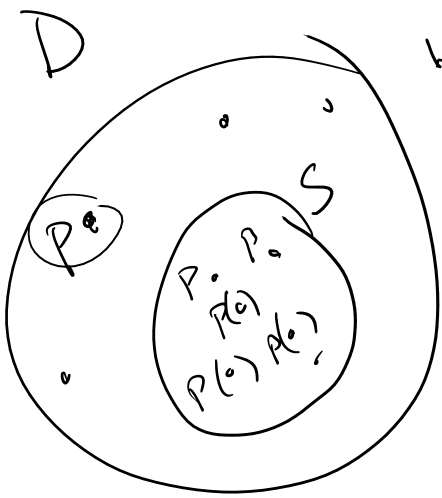
$$\forall n : \mathbb{Z}.\ Even(n) \rightarrow n = 2$$

$$\forall n : \mathbb{Z}.\ Even(n) \leftrightarrow n = 2$$

$\forall n:(\text{even integers}).\ Odd(n+1)$

$\cancel{\forall n:\mathbb{Z}.\ Odd(n+1)}$

✓ $\forall n:\mathbb{Z}.\ Even(n) \to Odd(n+1)$



want: $\forall x:S.\ P(x)$

$\equiv$

$\forall x:D.\ S(x) \to P(x)$

eg. Let $H(x) =$ "x attends Hendrix"

$C(x) =$ "x is cool"

"All Hendrix students are cool"

$\forall x:(\text{Hendrix students}).\ C(x)$

$\equiv$

$\forall x:\text{People}.\ H(x) \to C(x).$

"There is at least 1 cool Hendrix student"

$\exists x : (\text{Hendrix students}) .\ C(x)$

$\equiv$

$\exists x : \text{People} .\ H(x) \wedge C(x)$

in general, if S is a subdomain of D,

$$\boxed{\exists x : S .\ P(x) \equiv \exists x : D .\ S(x) \wedge P(x)}$$

eg. "Every even integer greater than 2 can be written as the sum of two prime numbers."

Assume $\text{Prime}(n)$ = "n is prime".

$\forall n : (\text{Even integers greater than } 2) .$
$\quad \exists p : \text{Primes} .\ \exists q : \text{Primes} .\ p + q = n .$

$\equiv \forall n : \mathbb{Z} .\ (\text{Even}(n) \wedge n > 2) \longrightarrow$
$\quad (\exists p : \mathbb{Z} .\ \text{Prime}(p) \wedge (\exists q : \mathbb{Z} .\ \text{Prime}(q) \wedge$
$\quad\quad (p + q = n)))$

eg. - $H(x) =$ x is a Hdx student

- $D(x) =$ x owns a dragon

$F(x,y) =$ x & y are friends.

"Every Hendrix student has a friend who owns a dragon".

$$\forall x : \text{People}.\ H(x) \to (\exists y : \text{People}.\ F(x,y) \land D(y))$$