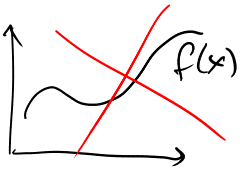
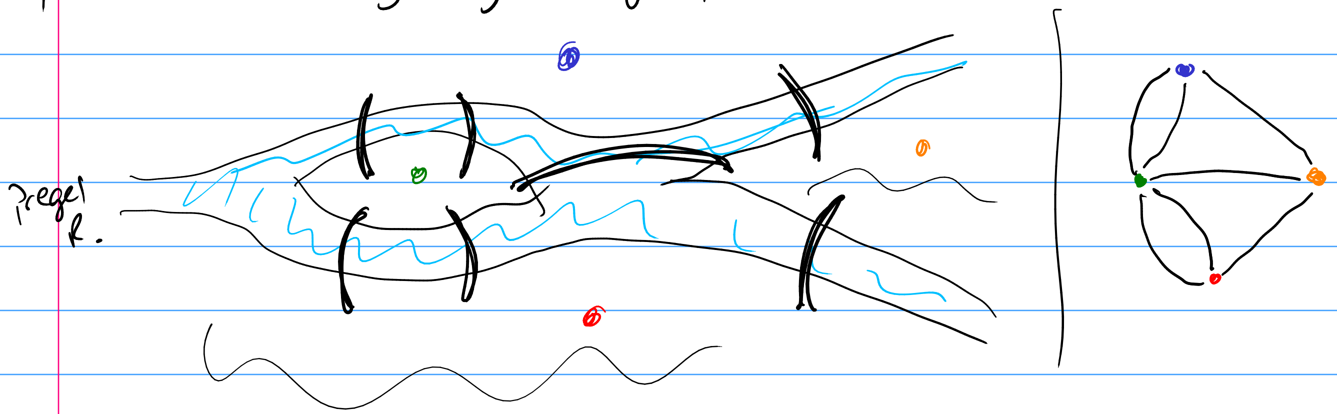
Graphs



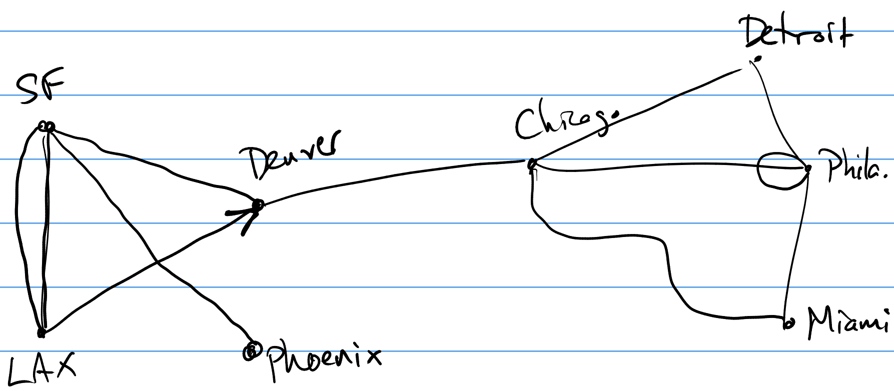
early 1700's: Königsberg bridge problem.



Leonhard Euler — 1736: solved the problem + invented graph theory.

Idea: Abstract away geography, keep only information about which land areas are connected to others.

Def'n A graph $G = (V, E)$ is a set of vertices V (aka nodes) and a set of edges E. Each edge connects two vertices, called its endpoints.



vertex ✓ vertice ✗

vertexes ✓ vertices ✓

Variants:

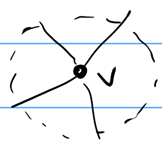
- Multiple edges between same 2 vertices.
- Self-loops, ie. edge from vertex to itself.
- Sometimes edges have a direction.
- Sometimes edges are labeled with some kind of weight/cost. (and/or vertices)

Examples — things we can model w/ graphs

- Classes — vertices are classes, edge = prerequisite.
  directed.
- Web pages + links — directed, self-loops ok, multiple edges ok.
- Rooms in a building, doors/hallways are edges
  — undirected (ie. edges bidirectional)
- Food chain — animals = vertices, who eats who = edge.
- Transportation networks — directed/undirected, edge weight = cost, time,
- Vertices = people, edges = relationship.

Def'n $u, v \in V$ are adjacent if they are the endpoints of an edge. (aka neighbors).

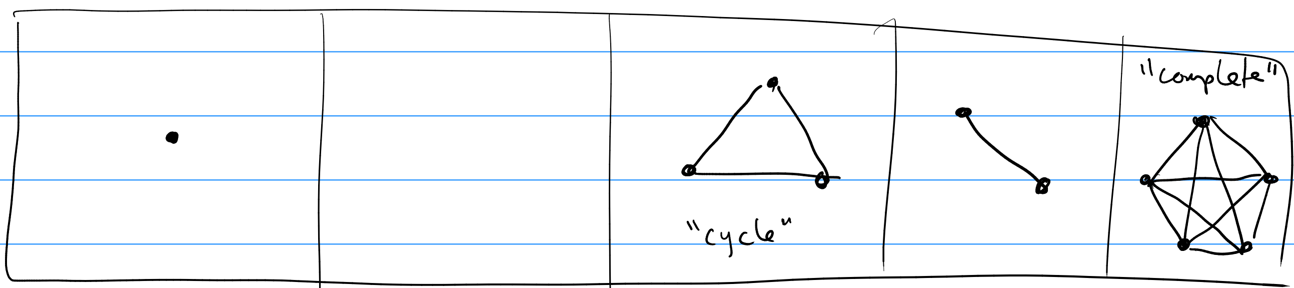
Def'n The degree of a vertex $v \in V$, written $\deg(v)$, is the number of edges adjacent to v. (loops count 2x).



— degree 4.

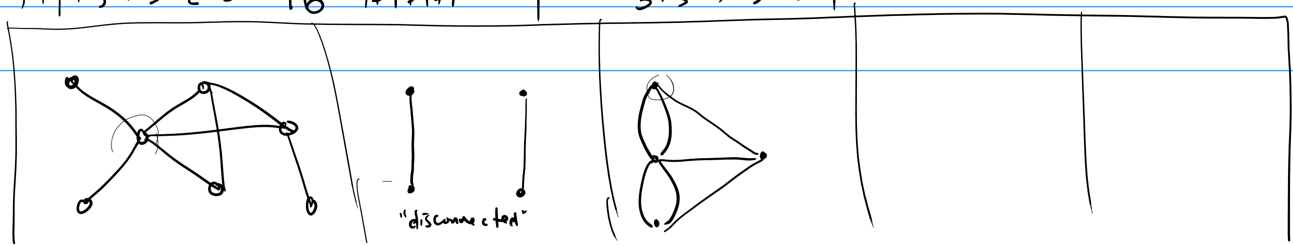
A degree-0 vertex connects to no edges.

Graph



Sum of degrees: 0 | 0 | 2+2+2 = 6. | 1+1 = 2 | 5·4 = 20.

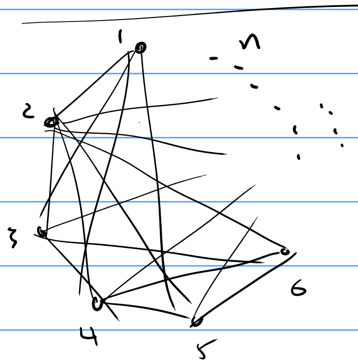
1+1+5+3+2+3+1 = 16 | 1+1+1+1 = 4 | 3+3+5+3 = 14



Handshake Lemma

In any undirected graph, the sum of all vertex degrees is twice the number of edges.

Proof. Each edge contributes 2 to the sum.



How many edges are in a complete graph w/ n vertices + all possible edges?

① $n \times (n-1)$ = degree sum, hence # edges = $\frac{n(n-1)}{2}$.

\# vertices   # of other vertices

② # of edges = # of pairs of vertices = $\binom{n}{2}$.

$$" \frac{n!}{2!(n-2)!} = \frac{n\cdot(n-1)\cdot}{2}$$

③ $(n-1) + (n-2) + (n-3) + \dots + 1$.

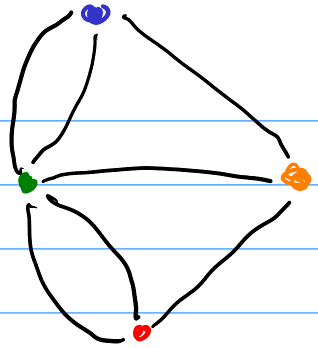
\# edges touching vertex 1   # so far uncounted edges @ vertex 2   etc.

$$1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Königsberg bridge problem, again.

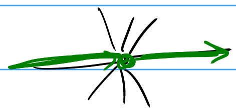
Q: Can you walk over all bridges, crossing each bridge exactly once?

Q: Is there a path (i.e. a sequence of connected edges) that contains every edge exactly once?



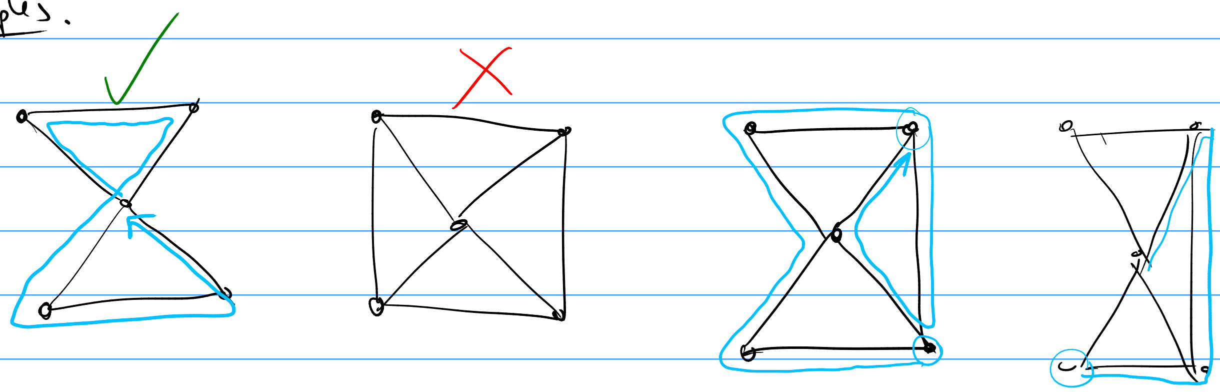
↳ Called an Eulerian path.

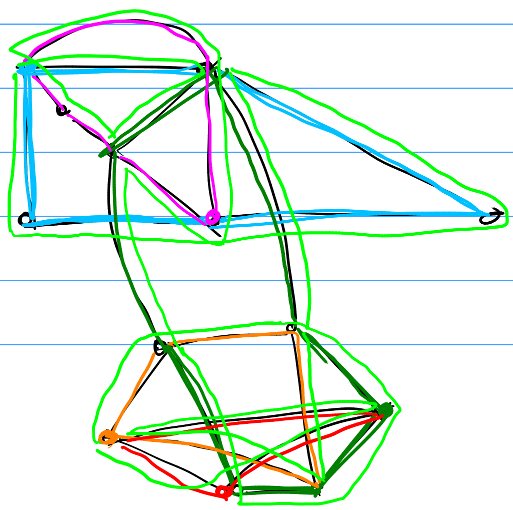
Any vertex w/ an odd degree must be either the start or end of an Eulerian path, since every time we visit a vertex in the middle of a path, it uses up 2 edges.
+ eventually an odd-degree vertex will be reduced to 1 edge — will get stuck there, or we must start there.



→ Any graph w/ more than 2 odd-degree vertices does not have an Eulerian path.

Examples.





Carl Hierholzer (1871):

If a graph has at most 2 odd-degree vertices, then it <u>does</u> have an Eulerian path.

<u>Def'n</u> An Eulerian <u>circuit</u> is an Eulerian path that begins + ends @ the same vertex.

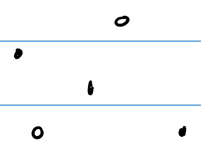
Thm (Euler + Hierholzer): A connected graph has an Eulerian <u>circuit</u> <u>iff</u> all vertices have even degree.

(Note: if we have graph w/ 2 odd vertices, just add an edge between them, use above theorem, then delete the edge to get Eulerian path).

<u>Proof</u>. ($\Rightarrow$) Euler; already proved.
($\Leftarrow$) Proof by algorithm (Hierholzer)

By induction on the # of edges.
Let G be an undirected graph, and suppose all vertices of G have even degree.

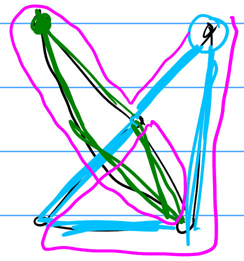
- Base case: G has 0 edges.
  i.e. G is a collection of vertices.
  Then G has an Eulerian circuit — just start @ any vertex + stay there.

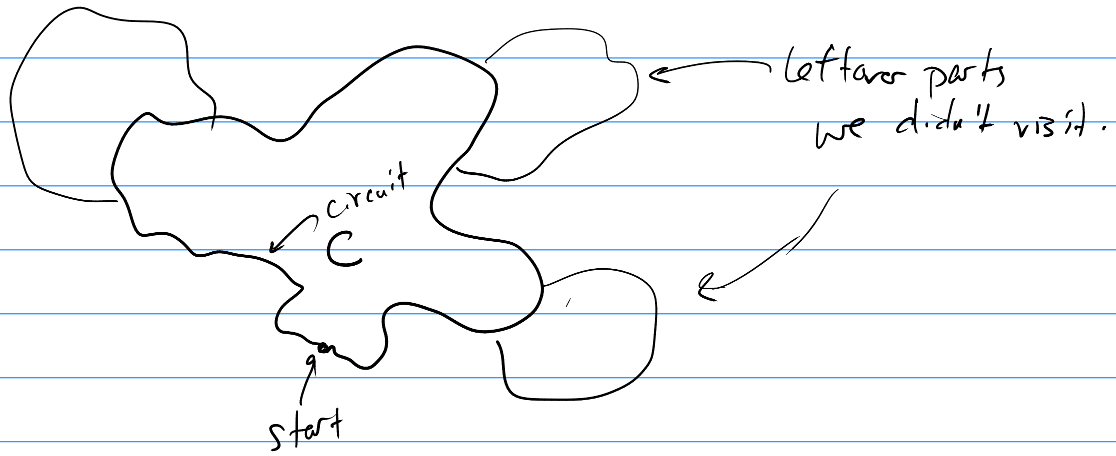


- Now let $k \in \mathbb{N}$, suppose any graph with $\leq k$ edges + all even degrees has an Eulerian circuit.

Let G be a graph w/ $k+1$ edges + suppose all vertices of G have even degree.
Pick any vertex of G and start walking / following edges.

But never repeat an edge. Claim: if you keep doing this until you get stuck, you will in fact get stuck @ the original starting vertex. Because every time you go through a vertex you use up 2 edges. So every vertex will always have an even # of unused edges — hence we can't get stuck there — except the starting vertex.





Call our circuit C. If we delete all edges of C from the graph, we will be left w/ some possibly disconnected pieces. Each piece has fewer edges, + all vertices still have even degree, since we deleted an even # of edges from each vertex. So by our induction hypothesis, each piece has an Eulerian circuit.
Now just "splice in" a circuit for each piece at a point where it touches the original circuit C. That will result in an Eulerian circuit for the entire graph.