Addition rule

Ex. You want to order lunch. Restaurant A has 13 items on its menu. Restaurant B has 24. They don't have any items in common. How many choices do you have for lunch?

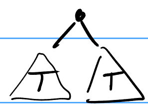
Addition rule: Suppose choosing something means either making one choice in c_1 ways or making a different choice in c_2 ways, and the choices do not overlap. Then the total # of choices is $c_1 + c_2$.

In set-theoretic terms: $|A \cup B| = |A| + |B|$ if A, B are disjoint, i.e. $A \cap B = \emptyset$.

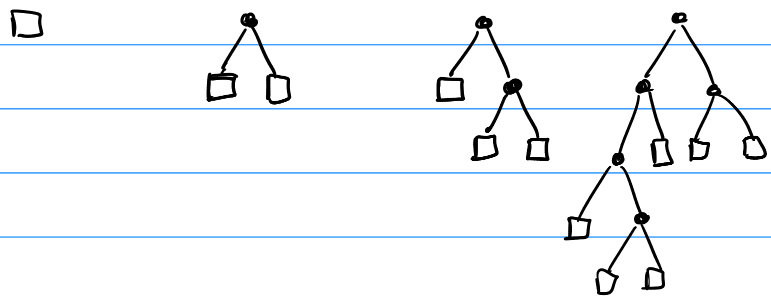
Ex.

A binary tree is either:

- A leaf, $\square$ or
- A branch node $\bullet$ with two binary trees as left + right children:

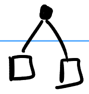
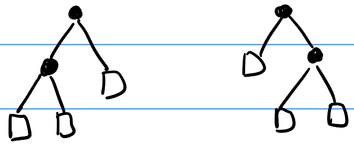
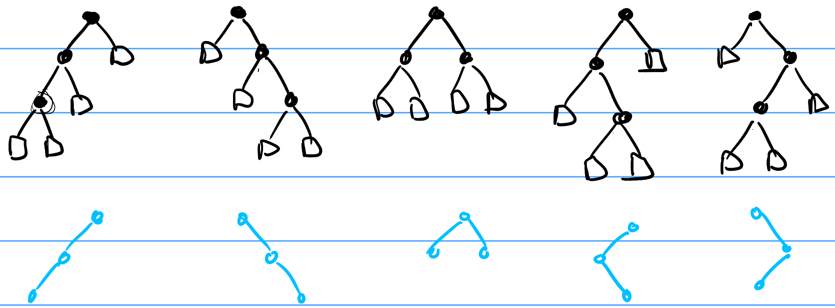
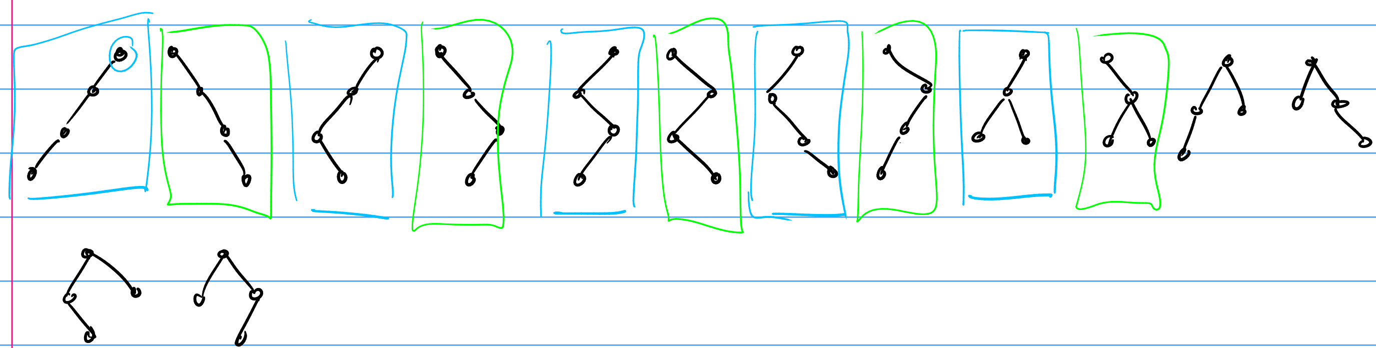


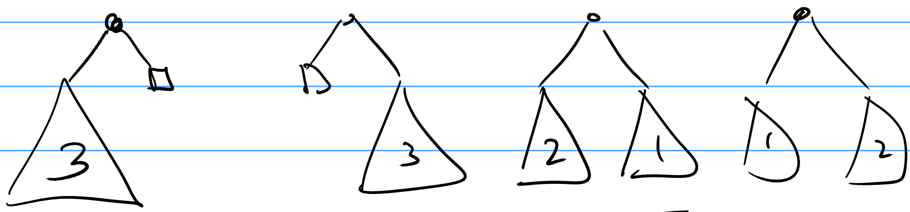
ex.



Q: How many different binary trees are there with 0 branch nodes? 1? 2? 3? ... n?

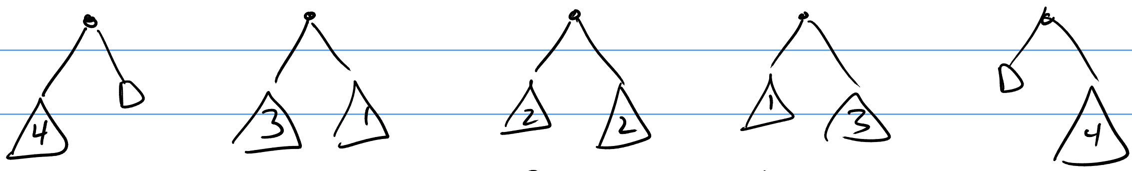
Let's try listing them.

# branch nodes	trees
0	
1	
2	
3	
4	



$5 + 5 + 2 \times 1 + 1 \times 2 = 14.$

5



14   5×1 2×2 1×5 1×14

$= 14 + 5 + 4 + 5 + 14 = 42.$

n	0	1	2	3	4	5	6	
# trees	1	1	2	5	14	42	132	...

Catalan numbers.

$$42 \times 1 + 14 \times 1 + 5 \times 2 + 2 \times 5 + 1 \times 14 + 1 \times 42$$
$$= 84 + 20 + 28 = 84 + 48 = 132$$

$T(n)$ = # of binary trees with n branch nodes.

$$T(n) = T(n-1) \times T(0) + T(n-2) \times T(1) + T(n-3) \times T(2) + \ldots$$

$$T(6) = T(5) \times T(0) + T(4) \times T(1) + \ldots$$

Subtraction rule (Principle of Inclusion-Exclusion / PIE)

If choosing something means making one choice in c_1 ways <u>OR</u> another choice in c_2 ways, but the choices overlap, the total # of choices is $c_1 + c_2 -$ (# choices in common).

$$|A \cup B| = |A| + |B| - |A \cap B|$$

<u>Ex.</u> Recall the restaurants from before with 13 and 24 menu items, respectively. Now suppose they have 9 items in common.

$$24 + 13 - 9 = 37 - 9 = 28.$$

<u>Ex.</u> How many strings of 8 bits either start w/ a 1 or end with 00?

By PIE/subtraction rule,

(start w/ 1) + # (end w/ 00) − # (both start w/ 1, end w/ 00).

$$= 2^7 + 2^6 - 2^5 = 128 + 64 - 32$$

$$= 128 + 32 = 160.$$