Strong induction.

Ex. Recall the Fibonacci sequence:

$F_0 = 0$

$F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$.

0, 1, 1, 2, 3, 5, 8, 13, ...

1 2 4 8 16 32 64 ..

Conjecture: $\forall n:\mathbb{N}.\ \underbrace{F_n \le 2^{n-1}}_{P(n)}$.

Proof (attempted). Let's prove by induction. We must prove $P(0) \wedge (\forall k:\mathbb{N}.\ P(k) \to P(k+1))$.

- $P(0)$ says $F_0 \le 2^{-1} \equiv 0 \le \frac{1}{2}$ ✓

- Now we must prove the induction step, $\forall k:\mathbb{N}.\ P(k) \to P(k+1)$.

So suppose k is an arbitrary natural # and suppose $P(k)$, that is, $\boxed{F_k \le 2^{k-1}}$. We must show $P(k+1)$, that is, $F_{k+1} \le 2^k$.

F_{k+1}

$=$ { def'n of Fibonacci #s } ✗

$F_k + F_{k-1}$

$\le$ { IH }

$2^{k-1} + F_{k-1}$

$F_n = F_{n-1} + F_{n-2}$ only if $n \ge 2$.

Strong induction.

Idea: when proving $P(k+1)$, we can assume $P(0), P(1), P(2), \ldots, P(k)$ are *all* true, not just $P(k)$.

Formally:

$$P(0) \wedge (\forall k:\mathbb{N}.\ \overbrace{[\forall j:\mathbb{N}.\ j \le k \to P(j)]}^{\text{Assume } P(0), P(1), \ldots, P(k) \text{ all true.}} \to P(k+1)) \longrightarrow (\forall n:\mathbb{N}.\ P(n))$$

Conjecture: $\forall n : \mathbb{N}.\ F_n \leq 2^{n-1}$.

Proof: by strong induction.

- $P(0) \equiv F_0 \leq 2^{-1}$ ✓
- $P(1) \equiv F_1 \leq 2^0 \equiv 1 \leq 1$ ✓
- Now let k be an arbitrary natural number $k \geq 1$. We must show $P(k+1)$ is true, if we suppose $P(0), P(1), \ldots, P(k)$ are true.

Need two base cases so the induction step can "get off the ground"

$P(0)$ $P(1)$ $P(2)$ $P(3)$

$$
\begin{aligned}
& F_{k+1} \\
= \quad & \{\text{defn of Fibonacci, since } k \geq 1.\} \\
& F_k + F_{k-1} \\
\leq \quad & \{P(k) \text{ and } P(k-1) \text{ are true}\}. \\
& 2^{k-1} + 2^{k-2} \\
< \quad & \{2^{k-2} < 2^{k-1}\} \\
& 2^{k-1} + 2^{k-1} \\
= \quad & \{\text{algebra}\} \\
& 2 \cdot 2^{k-1} \\
= \quad & \{\text{algebra}\} \\
& 2^k.
\end{aligned}
$$

Hence, $F_{k+1} \leq 2^k$.

Therefore, by strong induction, $F_n \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Note: strong + "weak" induction are actually logically equivalent.

Challenge: prove $\forall n : \mathbb{N}.\ (3/2)^n < F_n$.

Ex: Fundamental Theorem of Arithmetic: every $n \geq 1$ can be written uniquely as a product of primes.

We will just prove every integer ≥ 2 can be written as a product of primes.

Proof. Let $P(n)$ be the proposition "n can be written as a product of primes." We will prove $\forall n \geq 2.\ P(n)$ by (strong?) induction.

- $P(2)$ says "2 can be written as a prod. of primes": ✓ 2 is prime.

- Now let $k \geq 2$, and suppose $P(2), P(3), \ldots, P(k)$ are all true. We must show $P(k+1)$.
  - $k+1$ is either prime or composite.
  - if it is prime, then $P(k+1)$ is true. since $k+1$ is a product of itself.
  - Otherwise, there must be a, b such that $ab = k+1$ and $2 \leq a, b \leq k$. By our induction hypothesis, $P(a)$ and $P(b)$ are true, that is, a and b can be written as products of primes. Therefore so can $k+1 = ab$.

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