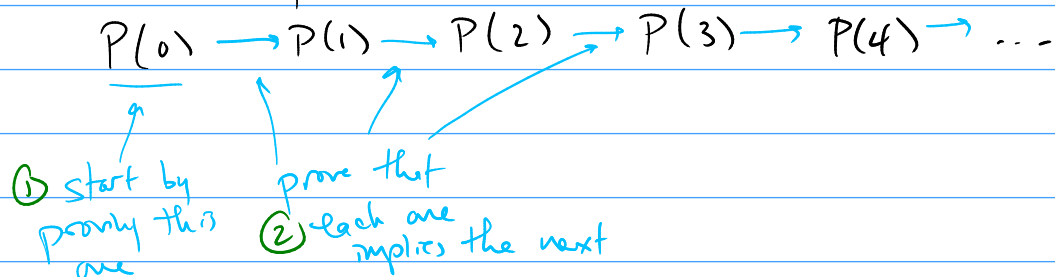


Induction.

A way to prove propositions of the form $\forall n \in \mathbb{N}. P(n)$.

Idea: want to prove



In formal propositional logic:

$$(P(0) \wedge (\forall k \in \mathbb{N}. P(k) \rightarrow P(k+1))) \longrightarrow (\forall n \in \mathbb{N}. P(n))$$

① "base case" ② "induction step"

Ex. $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

We proved this with pictures; let's prove it using induction.

Let $P(n) = "1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}"$; we wish to prove $\forall n \in \mathbb{N}. P(n)$.

Proof. By induction. The principle of induction tells us that we can conclude $\forall n \in \mathbb{N}. P(n)$ if we are able to prove $P(0) \wedge (\forall k \in \mathbb{N}. P(k) \rightarrow P(k+1))$. We will prove both separately.

• First, we will prove $P(0)$ ("base case").

$1 + 2 + \dots + 0$ means we don't add up any numbers, and an empty sum is 0. Also, $\frac{0 \cdot (0+1)}{2} = 0$. ✓

• (Optionally, to be sure, we could also show $P(1)$ - says $1 = \frac{1 \cdot (1+1)}{2} = 1$. ✓)

• Now we must show $\forall k: \mathbb{N}. P(k) \rightarrow P(k+1)$.

So let k be an arbitrary natural number, and

suppose $P(k)$ is true, that is,

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}. \quad \text{(KNOW) (induction hypothesis)}$$

we must show $P(k+1)$, that is,

$$1 + 2 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}. \quad \text{(WANT).}$$

So,

$$1 + 2 + \dots + (k+1) = \quad \{\text{rewrite}\}$$

$$= \underbrace{1 + 2 + \dots + k}_{\downarrow} + (k+1) \quad \{\text{IH}\}$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \quad \{\text{algebra}\}$$

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \quad \{\text{algebra}\}$$

$$\frac{(k+1)(k+2)}{2} \quad \checkmark$$

~~Q.E.D.~~

Since we showed $P(k) \rightarrow P(k+1)$ for arbitrary k ,

therefore $\forall k: \mathbb{N}. P(k) \rightarrow P(k+1)$,

Thus by principle of induction, $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ for all n .

Ex Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.

Let $P(n) = "2^n < n!"$.

Idea: prove $\forall n: \mathbb{N}. P(n)$.

$$P(0) : 2^0 < 0! \rightarrow 1 < 1 \quad \times$$

$$P(1) : 2^1 < 1! \rightarrow 2 < 1 \quad \times$$

$$P(2) : 2^2 < 2! \rightarrow 4 < 2 \quad \times$$

$$P(3) = 2^3 < 3! \rightarrow 8 < 6 \quad \times$$

$$P(4) = 2^4 < 4! \rightarrow 16 < 24 \quad \checkmark$$

$$P(5) = 2^5 < 5! \rightarrow 32 < 120 \quad \checkmark$$

So actually, $P(n)$ is not true for all \mathbb{N} , but we conjecture it is true for $n \geq 4$.

We can still prove this by induction, we just start at 4 instead of 0.

Proof. We must show $P(4) \wedge (\forall k: \mathbb{N}. (k \geq 4) \wedge P(k) \rightarrow P(k+1))$.

- $P(4) = "2^4 < 4!" \rightarrow 16 < 24 \quad \checkmark$.

- Now let k be an arbitrary natural number ≥ 4 and suppose $P(k)$, that is, $2^k < k!$. We must show $P(k+1)$, that is, $2^{k+1} < (k+1)!$.

$$\begin{aligned} & 2^{k+1} \\ &= \frac{2 \cdot 2^k}{2 \cdot 2^k} \quad \{\text{algebra}\} \\ &< \frac{2 \cdot k!}{2 \cdot k!} \quad \{\text{IH, } 2^k < k!\} \\ &< \frac{2 \cdot k!}{(k+1) \cdot k!} \quad \{2 < k+1, \text{ since } k \geq 4\} \\ &= \frac{2 \cdot k!}{(k+1) \cdot k!} \quad \{\text{algebra}\} \\ &= \frac{2}{k+1} \end{aligned}$$

Hence $2^{k+1} < (k+1)!$. \downarrow