Induction.

A way to prove propositions of the form $\forall n : \mathbb{N}.\ P(n)$.

Idea: want to prove

$$P(0) \longrightarrow P(1) \longrightarrow P(2) \longrightarrow P(3) \longrightarrow P(4) \longrightarrow \ldots$$

① start by proving this one

prove that ② each one implies the next

In formal propositional logic:

induction hypothesis.

$$\Big(\underbrace{P(0)}_{①\ \text{"base case"}} \wedge \underbrace{\big(\forall k : \mathbb{N}.\ P(k) \rightarrow P(k+1)\big)}_{②\ \text{"induction step"}}\Big) \longrightarrow \big(\forall n : \mathbb{N}.\ P(n)\big)$$

Ex. $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

We proved this with pictures; let's prove it using induction.

Let $P(n) =$ "$1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$"; we wish to prove $\forall n : \mathbb{N}.\ P(n)$.

Proof. By induction. The principle of induction tells us that we can conclude $\forall n : \mathbb{N}.\ P(n)$ if we are able to prove $P(0) \wedge (\forall k : \mathbb{N}.\ P(k) \rightarrow P(k+1))$. We will prove both separately.

- First, we will prove $P(0)$ ("base case").
  $1+2+\ldots+0$ means we don't add up any numbers, and an empty sum is 0. Also, $\dfrac{0 \cdot (0+1)}{2} = 0$. ✓.
- (Optionally, to be sure, we could also show $P(1)$ — says $1 = \dfrac{1 \cdot (1+1)}{2} = 1$. ✓)

- Now we must show $\forall k:\mathbb{N}.\ P(k) \to P(k+1)$. So let k be an arbitrary natural number, and suppose $P(k)$ is true, that is,

$$1+2+\dots+k = \frac{k(k+1)}{2}.$$ (KNOW) (induction hypothesis (IH))

We must show $P(k+1)$, that is,

$$1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2}.$$ (WANT).

So,

$$
\begin{aligned}
& 1+2+\dots+(k+1) \\
= & \quad \{\text{rewriting}\} \\
& \underbrace{1+2+\dots}+k+(k+1) \\
= & \quad \{\text{IH}\} \\
& \frac{k(k+1)}{2} + (k+1) \\
= & \quad \{\text{algebra}\} \\
& \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
= & \quad \{\text{algebra}\}. \\
& \frac{(k+1)(k+2)}{2} \quad \checkmark
\end{aligned}
$$

Since we showed $P(k) \to P(k+1)$ for arbitrary k, therefore $\forall k:\mathbb{N}.\ P(k) \to P(k+1)$,

Thus by principle of induction, $1+2+\dots+n = \frac{n(n+1)}{2}$ for all n.

Ex: Recall $n! = 1\cdot 2\cdot 3\cdot \dots \cdot n$.

Let $P(n) =$ "$2^n < n!$".

Idea: prove $\forall n:\mathbb{N}.\ P(n)$.

$P(0)$: $2^0 < 0! \longrightarrow 1 < 1$ ✗

$P(1)$: $2^1 < 1! \longrightarrow 2 < 1$ ✗

$P(2)$: $2^2 < 2! \longrightarrow 4 < 2$ ✗

$P(3) : 2^3 < 3! \to 8 < 6$ ×
$P(4) : 2^4 < 4! \to 16 < 24$ ✓
$P(5) : 2^5 < 5! \to 32 < 120$ ✓

So actually, $P(n)$ is not true for all $\mathbb{N}$, but we conjecture it is true for $n \geq 4$.

We can still prove this by induction, we just start at 4 instead of 0.

Proof. We must show $P(4) \wedge (\forall k : \mathbb{N}.\ (k \geq 4) \wedge P(k) \to P(k+1))$.

- $P(4) =$ "$2^4 < 4!$" $\to 16 < 24$ ✓.
- Now let k be an arbitrary natural number ≥ 4 and suppose $P(k)$, that is, $2^k < k!$. We must show $P(k+1)$, that is, $2^{k+1} < (k+1)!$.

$$
\begin{aligned}
& 2^{k+1} \\
= \quad & \{\text{algebra}\} \\
& 2 \cdot 2^k \\
< \quad & \{\text{IH, } 2^k < k!\} \\
& 2 \cdot k! \\
< \quad & \{2 < k+1, \text{ since } k \geq 4\} \\
& (k+1) \cdot k! \\
= \quad & \{\text{algebra}\} \\
& (k+1)!
\end{aligned}
$$

Hence $2^{k+1} < (k+1)!$.