Sequences

Examples

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$5, 5, 5, 5, 5, \ldots$

$2, 6, 18, 54, 162, \ldots$ (geometric sequence — multiply by same value each time)

$2, 5, 8, 11, 14, \ldots$ (arithmetic sequence — add same value each time)

Def'n A sequence is a function from a contiguous subset of $\mathbb{N}$ to a set S. We use subscript notation like a_n to denote the n^{th} term of the sequence.

In other words, a sequence is a list of values where each value has an index.

eg $a_1 = 1, \; a_2 = \frac{1}{2}, \; a_3 = \frac{1}{3}, \ldots \; a_n = \frac{1}{n}$.

eg. The string "hello" can be thought of as a finite sequence $s_0 = $ 'h', $s_1 = $ 'e', etc

closed form — defines a_n in terms of n.

eg. $a_0 = 2, \; a_1 = 5, \; a_2 = 8, \ldots$ $\boxed{a_n = 2 + 3n}$

OR $\boxed{\begin{array}{l} a_n = 3 + a_{n-1}. \\ a_0 = 2. \end{array}}$ $\longrightarrow$ on its own, many solutions
eg. $1, 4, 7, 10 \ldots$
$-12, -9, -6, \ldots$

recurrence — defines a_n in terms of previous a_i.

eg. $2, 6, 18, 54, 162, \dots$

$$a_n = 3a_{n-1}$$
$$a_0 = 2.$$

One way to try to find a closed form: keep substituting the recurrence and look for a pattern.

$$\begin{aligned} a_n &= 3a_{n-1} \\ &= 3(3a_{n-2}) \\ &= 3(3(3a_{n-3})) \\ &= 3^{3} \cdot a_{n-3} \\ &= 3^k \cdot a_{n-k} \\ &= 3^n \cdot a_0 = \boxed{2 \cdot 3^n} \end{aligned}$$

— closed form.

Ex. $\boxed{\begin{aligned} a_n &= 2a_{n-1} + 1 \\ a_0 &= 0. \end{aligned}}$

n	0	1	2	3	4	5	
a_n	0	1	3	7	15	31	...

Look for patterns, conjecture that $a_n = 2^n - 1$

Or substitute:

$$\begin{aligned} a_n &= 2a_{n-1} + 1 \\ &= 2(2a_{n-2} + 1) + 1 \\ &= 4a_{n-2} + 2 + 1 \\ &= 4(2a_{n-3} + 1) + 2 + 1 \\ &= 8a_{n-3} + 4 + 2 + 1 \\ &\dots ? \end{aligned}$$

How can we know that $a_n = 2^n - 1$ is a valid closed form?

Easy! Just check that it satisfies the recurrence.

- $a_0 = 0$? $\quad a_0 = 2^0 - 1 = 1 - 1 = 0$ ✓
- $a_n = 2a_{n-1} + 1$? $\quad$ Substitute $a_n = 2^n - 1$ and check that the equation is true

$$2^n - 1 \stackrel{?}{=} 2(2^{n-1} - 1) + 1$$
$$2^n - 1 \stackrel{?}{=} 2^n - 2 + 1 = 2^n - 1 \quad ✓$$

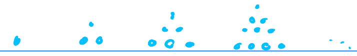
Ex.

(a) 1, 3, 5, 7, 9, 11, ...

$a_n = a_{n-1} + 2$ $\qquad$ $a_n = 2n+1.$

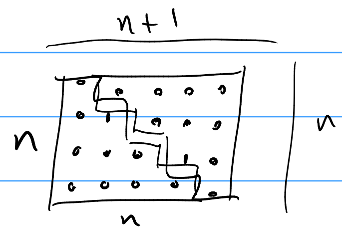
$a_0 = 1.$

(b) 0, 1, 3, 6, 10, 15, ... Triangular #'s:



$a_n = a_{n-1} + n.$

$a_0 = 0.$



$$\frac{n(n+1)}{2} = a_n.$$

(c) 0, 1, 1, 2, 3, 5, 8, 13, 21, ... Fibonacci #'s

$F_n = F_{n-1} + F_{n-2}$

$F_0 = 0$

$F_1 = 1$

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \hat{\varphi}^n\right)$$

where

$$\varphi, \hat{\varphi} = \frac{1 \pm \sqrt{5}}{2}.$$