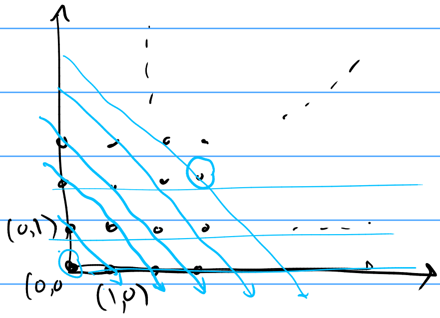
Recall: set A is countably infinite if $|A| = |\mathbb{N}|$, that is, there is a bijection (invertible function) $f: A \to \mathbb{N}$.

- We saw that countably infinite sets include $\mathbb{N}$, $2\mathbb{N}$, $\mathbb{Z}$.

Q: are there any sets which are not countable???

- $\mathbb{N} \times \mathbb{N}$ — pairs of natural #'s. Is this countable?



Matching w/ $\mathbb{N}$?

$\mathbb{N} \times \mathbb{N}$	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	...		
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	...		
$\mathbb{N}$	0	1	2	3	4	5	6	

✗ doesn't work

List them by diagonals!

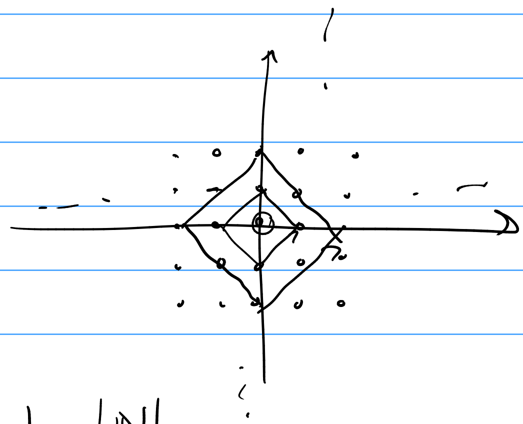
✓

$\mathbb{N} \times \mathbb{N}$	(0,0)	(0,1)	(1,0)	(0,2)	(1,1)	(2,0)	(0,3)	(1,2)	(2,1)	(3,0)...		
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	...						
$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9...		

So $\mathbb{N} \times \mathbb{N}$ is also countable!

What about $\mathbb{Z} \times \mathbb{Z}$?

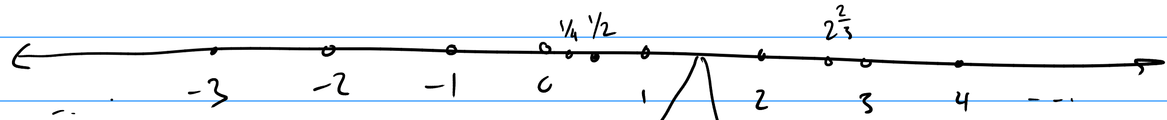
Yes! List by diamonds:



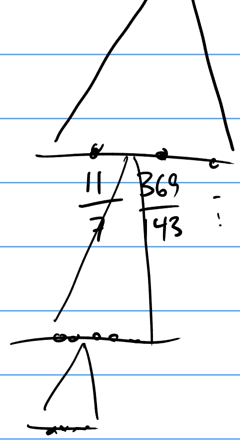
Or, since $|\mathbb{Z}| = |\mathbb{N}|$,

therefore $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Consider $\mathbb{Q}$.



Is $\mathbb{Q}$ countable?



Surprisingly, yes!
Rational is really just a pair of integers,
and we already know $\mathbb{Z} \times \mathbb{Z}$ is countable.

Consider the set $\mathbb{R}$ of real numbers.
Actually, consider $\{x \mid x \in \mathbb{R},\ 0 \leq x < 1\}$.
Write them as infinite decimal expansions:

$$0.d_1 d_2 d_3 d_4 d_5 \ldots$$

eg.

$$0.10000000\ldots$$

$$0.3333333\ldots$$

$$0.129637840772\ldots$$

How do we prove a set is <u>not</u> countable?

$$\neg \mathrm{Countable}(\mathbb{R})$$

$$\equiv \qquad \{\text{def}^n\}$$

$$\neg(\exists f : \mathbb{N} \to \mathbb{R}.\ \mathrm{Bijection}(f))$$

$$\equiv \qquad \{\text{def}^n\}$$

$$\neg(\exists f : \mathbb{N} \to \mathbb{R}.\ \underbrace{\mathrm{Injection}(f)}_{(1-1)} \wedge \underbrace{\mathrm{Surjection}(f)}_{(\text{onto})})$$

$$\equiv \qquad \{\text{de Morgan}\}$$

$$\forall f : \mathbb{N} \to \mathbb{R}.\ \neg \mathrm{Injection}(f) \vee \underline{\neg \mathrm{Surjection}(f)}.$$

Note: a function $\mathbb{N} \to \mathbb{R}$ is like an infinite list of real #s.

n	0	1	2	3
$f(n)$	$1/2$	π	$0.2983\ldots$	

So we will show <u>any</u> such function is <u>not</u> onto,
that is, any list of real #'s will necessarily leave some out.

So let f be an arbitrary $\mathbb{N} \to \mathbb{R}$ function

n	$f(n)$	
0	$0.d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15} \ldots$	$\longrightarrow$ pick $a_1 \neq d_{11}$.
1	$0.d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25} \ldots$	$\longrightarrow$ pick $a_2 \neq d_{22}$.
2	$0.d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35} \ldots$	$\longrightarrow$ pick $a_3 \neq d_{33}$.
3	$0. \ldots$	
4		

Given this list, create a real # as follows:

$$0.a_1 a_2 a_3 \ldots$$

This number is nowhere on the list!
Any list of real #'s must leave some out.
$\longrightarrow$ $\mathbb{R}$ is "more infinite" than $\mathbb{N}$!