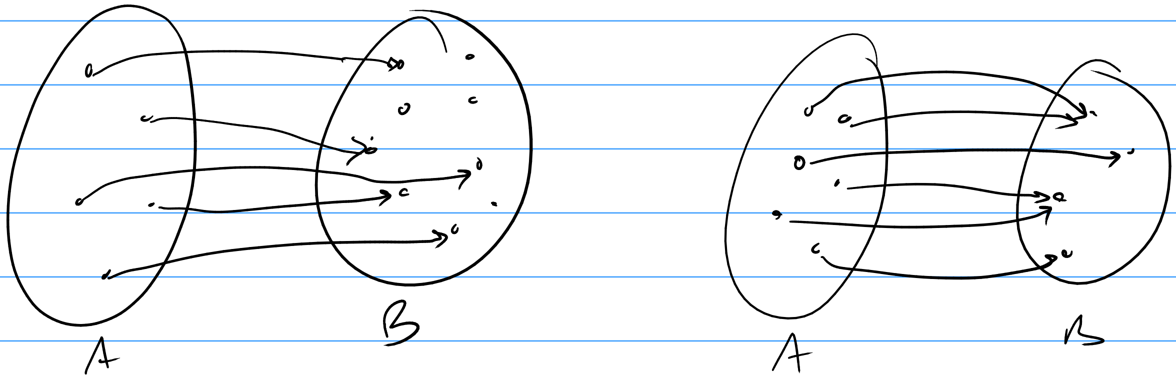
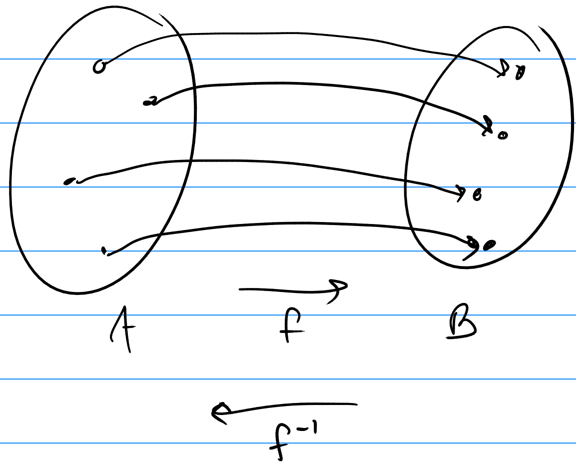
1-1 (injective) onto (surjective).



both 1-1 and onto
(bijective / bijection)



Def'n A function $f: A \to B$ which is both 1-1 and onto is called a bijection (bijective), or invertible. We write $f^{-1}: B \to A$ ("f inverse") for the unique function such that $f(a) = b$ iff $f^{-1}(b) = a$.

⚠ $f^{-1} \neq \frac{1}{f}$.

Ex. Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$. Is f invertible?

Yes, it is! Proof:

- f is 1-1. We must show that for all $x, y : \mathbb{Z}$, $f(x) = f(y) \to x = y$. Let x, y be arbitrary integers.

  $f(x) = f(y)$

  $\to$ {def'n of f}

$$x+1 = y+1$$

$$\rightarrow x = y. \quad \checkmark \quad \{\text{algebra}\}$$

- f is onto, we must show $\forall z : \mathbb{Z}$, there exists $y \in \mathbb{Z}$. such that $f(y) = z$.

  Let $z \in \mathbb{Z}$ be arbitrary. $z-1 \in \mathbb{Z}$, and
  $f(z-1) = (z-1)+1 = z.$ ✓

Ex. Show $g : \mathbb{Q} \to \mathbb{Q}$, $g(x) = 3x+2$ is invertible.

Thm. $f : A \to B$ is invertible iff it has an inverse, that is, there exists a function $g : B \to A$ such that
(1) $\forall a \in A.\ g(f(a)) = a.$
(2) $\forall b \in B.\ f(g(b)) = b.$

Proof : Challenge!

This means we have 2 different ways to show f is invertible:

(1) Show it is 1-1 and onto.
(2) Show it has an inverse (in both directions).

Ex. Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$ as before. Is f invertible?
Yes! Proof: $g : \mathbb{Z} \to \mathbb{Z}$, $g(x) = x-1$.

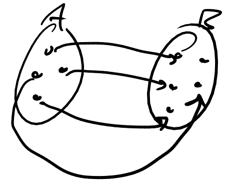
- $f(g(x)) = f(x-1) = (x-1)+1 = x$ ✓
- $g(f(x)) = g(x+1) = (x+1)-1 = x$ ✓

Defn Let A, B be sets. If there exists a bijection $f : A \to B$, then we say that A and B have the same cardinality, and we write

$$|A| = |B|.$$

Note, this works for infinite sets too!

Also, if there is an injection (1-1) $f : A \to B$, we say $|A| \leq |B|$.



<u>Def'n</u> A set A is <u>countable</u> if it has the same cardinality as a subset of $\mathbb{N}$.

Note that any finite set is countable. If $|A| = |\mathbb{N}|$, then we say A is <u>countably infinite</u>.

Ex. $C = \{\text{blue}, \text{red}, \text{green}\}$ is countable:

$$\begin{array}{cccc} C: & \text{blue} & \text{red} & \text{green} \\ & \updownarrow & \updownarrow & \updownarrow \\ \mathbb{N}: & 0 & 1 & 2 \end{array}$$

<u>Ex.</u> $\mathbb{N}$ is countable

$$\begin{array}{ccccccc} \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & \cdots \end{array}$$

<u>Ex</u> $2\mathbb{N} = \{0, 2, 4, 6, 8, \dots\}$?

$$\begin{array}{ccccccc} 2\mathbb{N}: & 0 & 2 & 4 & 6 & 8 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & \cdots \end{array}$$

$$\left[\begin{array}{ll} f : \mathbb{N} \to 2\mathbb{N} & g : 2\mathbb{N} \to \mathbb{N} \\ f(n) = 2n & g(n) = n/2 \end{array}\right] \quad \text{inverse!}$$

Therefore, $|2\mathbb{N}| = |\mathbb{N}|$.

Ex. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is this countable?

$$\begin{array}{lcccccccccl}
\mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\
\mathbb{N}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots
\end{array}$$

$$|\mathbb{Z}| = |\mathbb{N}| \; !$$