What is a function?

- Takes input → output.

Examples:

$f(x) = x + 5$

$f(x) = 5$

$\sin(x)$

$\log(x)$

append : String × String → String

format: $\mathbb{Z}$ → String

bool2Int : Bool → $\mathbb{Z}$

Non-examples:

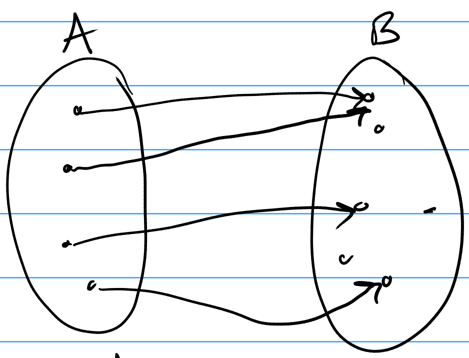
printgreeting()

$x = 2$

$f(x) = \sqrt{x}$ — returns 2 outputs for 1 input.

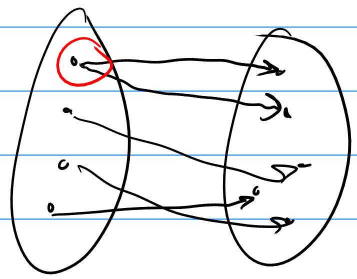
Defn Let A and B be sets. A function f from A to B, written $f : A \to B$, is an assignment of exactly one element of B (output) to every element of A (input).

A is the domain, B is the codomain. We write $f(a) = b$ to denote the particular b assigned to the input a.

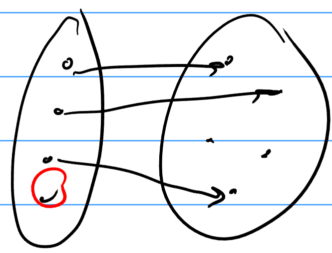


A function!

Not a function.



e.g. $f(x) = \sqrt{x}$.



Not a function.

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = 3/x$

Not defined on 0.

$f : (\mathbb{R} - \{0\}) \to \mathbb{R}$ ✓ OK!

Def'n The *range* of a function is the set of possible outputs:

$$\text{Rng}(f) = \{ b \mid b \in B,\ \exists a \in A.\ f(a) = b \}.$$

Eg. $f : \mathbb{N} \to \mathbb{N}$:   domain $= \mathbb{N}$

$f(n) = n^2$ codomain $= \mathbb{N}$

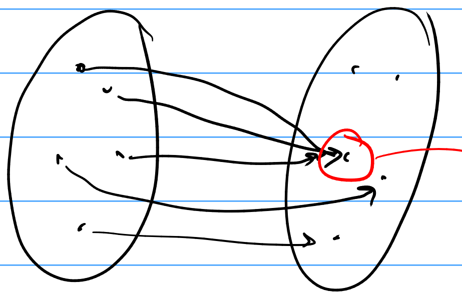
range $= \{0, 1, 4, 9, 16, \ldots\}$.

$f : \mathbb{Z} \to \mathbb{Z}$

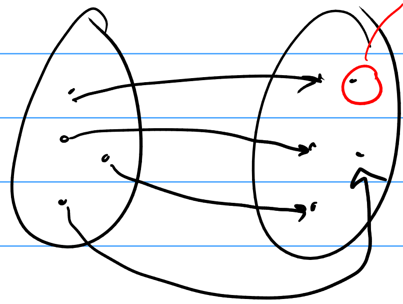
$f(n) = n^2 - 1$ codomain $\neq$ range

Q: When is it possible to turn a function into an "opposite function" that undoes the original function?

What could go wrong?



multiple outputs for same input.



element of codomain not in range.

eg. $f(x) = 5$ eg. $f : \mathbb{N} \to \mathbb{N}$

$f(n) = n^2$

Turns out there are only 2 possible problems!

Defn. An *injective* (1–1) function is a function where every input corresponds to its own, unique output. ie. no 2 inputs correspond to the same output.

Formally, given $f : A \to B$,

$$\forall a_1 \in A.\ \forall a_2 \in A.\ a_1 \neq a_2 \to f(a_1) \neq f(a_2).$$

Or, contrapositively,

$$\forall a_1 \in A.\ \forall a_2 \in A.\ f(a_1) = f(a_2) \rightarrow a_1 = a_2.$$

this version is nicer for proving things.

How do we show something is <u>not</u> 1-1?

$$\neg(\forall a_1 \in A.\ \forall a_2 \in A.\ f(a_1) = f(a_2) \rightarrow a_1 = a_2.)$$
$$\equiv \exists a_1 \in A.\ \exists a_2 \in A.\ \neg(f(a_1) = f(a_2) \rightarrow a_1 = a_2))$$
$$\equiv \exists a_1 \in A.\ \exists a_2 \in A.\ \neg((f(a_1) \neq f(a_2)) \vee (a_1 = a_2))$$
$$\equiv \exists a_1 \in A.\ \exists a_2 \in A.\ f(a_1) = f(a_2) \wedge a_1 \neq a_2.$$

ie. Show two elements of domain which are different but yield the same result when put into the function.

<u>Ex.</u> Prove $j : \mathbb{Z} \rightarrow \mathbb{Z}$, $j(x) = 3x + 2$ is 1-1.

<u>Proof.</u> We must show $\forall z_1 : \mathbb{Z}.\ \forall z_2 : \mathbb{Z}.\ j(z_1) = j(z_2) \rightarrow z_1 = z_2.$

Let z_1, z_2 be arbitrary integers.

- Suppose $j(z_1) = j(z_2)$. We must show $z_1 = z_2$.
  - $j(z_1) = j(z_2)$

    $\rightarrow$

    $3z_1 + 2 = 3z_2 + 2$

    $\rightarrow$ {algebra}

    $z_1 = z_2$.

<u>Ex.</u> Show that $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x^2 + 5$ is not injective.

<u>Proof</u> $-1 \neq 1$, but $f(-1) = f(1) = 6$.

Def'n A function $f : A \to B$ is <u>onto</u> / <u>surjective</u> / a <u>surjection</u> if "every element of B is hit", that is,

$$\underline{\forall b \in B.\ \exists a \in A.\ f(a) = b.}$$

To show something is <u>not</u> onto:

$$\neg(\forall b \in B.\ \exists a \in A.\ f(a) = b)$$
$$\equiv \underline{\exists b \in B.}\ \underline{\forall a \in A.\ f(a) \neq b.}$$

Ex. Show that $j : \mathbb{Z} \to \mathbb{Z}$, $j(x) = 3x + 2$ is <u>not</u> onto.

Proof. We must show there exists $y \in \mathbb{Z}$ which is not possible to get as the output of j.

- Consider $y = 6$. We must show for every $x \in \mathbb{Z}$, $j(x) \neq 6$.
  - Use a proof by contradiction. Suppose there does exist x such that $j(x) = 6$.

$$j(x) = 6$$
$$\to \quad \{\text{def'n of } j\}$$
$$3x + 2 = 6$$
$$\to$$
$$3x = 4$$
$$\to$$
$$x = \frac{4}{3}. \quad \lightning$$

contradiction symbol.

This is a contradiction, since we assumed x was an integer.

Ex Prove $k : \mathbb{Q} \to \mathbb{Q}$, $k(x) = 3x + 2$ is onto.

Proof. We must show $\forall x : \mathbb{Q}.\ \exists y : \mathbb{Q}.\ k(y) = x$.

Let x be an arbitrary rational. We must show there is some rational y such that $k(y) = x$.

$h(y) = x$

$\rightarrow 3y + 2 = x$

$\rightarrow y = \frac{x-2}{3}.$ $\leftarrow$ this is rational if x is.