Set operations

Def'n The cardinality of a finite set is the number of elements it contains (a natural #). The cardinality of a set A is written $|A|$.

Ex. $|\{1, 2, 7\}| = 3$

$|\{2, 4, \ldots, 100\}| = 50$

$|\emptyset| = 0.$

Cartesian product (Descartes)

Def'n If A and B are sets, the Cartesian product of A and B, written $A \times B$, is the set of all possible ordered pairs with the first element from A and the second from B. That is,

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Example. $A = \{1,2,3\}$ $B = \{\square, \Delta\}$

$$A \times B = \{(1,\square), (1,\Delta), (2,\square), (2,\Delta), (3,\square), (3,\Delta)\}.$$

We can think of this in terms of a grid w/ rows labelled by A and columns by B.

	$\square$	Δ
1	$(1,\square)$	$(1,\Delta)$
2	$(2,\square)$	$(2,\Delta)$
3	$(3,\square)$	$(3,\Delta)$

Hence

$$|A \times B| = |A| \times |B|$$

(for finite sets).

Power set

Def'n The power set of a set A, written $\mathcal{P}(A)$, is the set of all subsets of A.

Ex. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\{1,2\}, \{1\}, \varnothing, \{1,2,3\}, \{2\}, \{3\}, \{2,3\}, \{1,3\}\}$$

How do we systematically list all subsets? → Independently choose whether each element is in or out.

1?	2?	3?	subset
T	T	T	$\{1,2,3\}$
T	T	F	$\{1,2\}$
T	F	T	$\{1,3\}$
T	F	F	$\{1\}$
F	T	T	$\{2,3\}$
F	T	F	$\{2\}$
F	F	T	$\{3\}$
F	F	F	$\varnothing$

# of subsets doubles every time we add one more element.

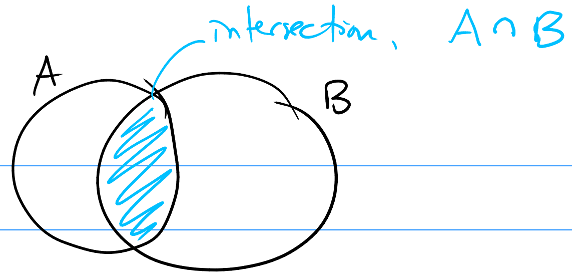
Hence, for finite sets, $|\mathcal{P}(A)| = 2^{|A|}$.

Does this work for sets of size 1? empty set?

$\mathcal{P}(\{1\}) = \{\{1\}, \varnothing\}$. $\quad |\mathcal{P}(\{1\})| = 2^1$

$\mathcal{P}(\varnothing) = \{\varnothing\}$ $\quad |\mathcal{P}(\varnothing)| = 1 = 2^0$.

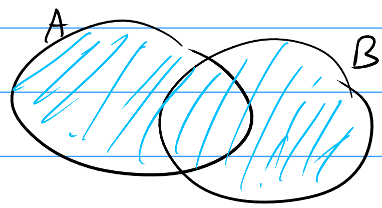
Intersection



<u>Def'n</u> The <u>intersection</u> of A and B, written $A \cap B$, is the set of elements A and B have in common

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}.$$

Union



<u>Def'n</u> The <u>union</u> of A and B, written $A \cup B$, is the set of elements which are in either set.

$$A \cup B = \{ x \mid x \in A \vee x \in B \}.$$

What can we say about $|A \cup B|$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(Principle of Inclusion - Exclusion (PIE))

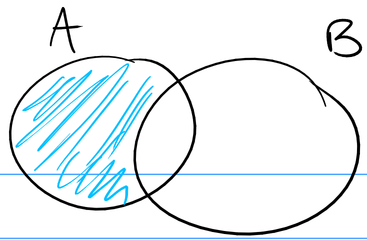
$$A \cap B \subseteq A \cup B$$

$$|A \cap B| \leq |A \cup B|$$

$$(A \cup B = A \cap B) \rightarrow (A = B)$$

$$\max(|A|, |B|) \leq |A \cup B| \leq |A| + |B|$$

Difference



<u>Def'n</u> The <u>difference</u> $A - B$ is all the elements of A that are not in B.

$$A - B = \{ x \mid x \in A \wedge x \notin B \}.$$