# Set Theory.

- Both practically very useful
- AND theoretically very important — historically basic theory on which a lot of mathematics has been built.

Def'n A set is an unordered, finite or infinite collection of objects, called elements or members of the set. Sets cannot contain the same object more than once, ie. the only thing that matters about a particular object is whether it is an element of the set or not.

We typically use capital letters to stand for sets, and lowercase to stand for elements.

The notation $x \in S$ means "$x$ is an element of $S$".
$x \notin S$ is an abbreviation for $\neg(x \in S)$.

## Writing sets

Two basic ways.

- List all the elements. We use curly braces + commas to separate elements, eg.

  $S = \{3, 5, 19, 24\}$

  $T = \{\pi, 62.9, -187\}$

  $\emptyset = \{\}$ empty set

  Can also use ... as abbreviation.

  $\{1, 2, 3, \ldots, 100\}$

  $\{1, 3, 5, \ldots, 99\}$

  etc.

"such that"

- Use "set builder notation": $\{\text{expression} \mid \text{conditions}\}$.

e.g. $\{x \mid x \in \mathbb{N},\ 1 \le x \le 100\}$

$\{3x^2 + 7 \mid x \in \mathbb{N},\ 1 \le x \le 100\}$

$\hookrightarrow = \{3 \cdot 1^2 + 7,\ \underline{3 \cdot 2^2 + 7},\ 3 \cdot 3^2 + 7, \ldots\}$

$\{3x^2 + 7 \mid x \in \mathbb{N},\ 1 \le x \le 100,\ \mathrm{Odd}(x)\}$

$\hookrightarrow = \{3 \cdot 1^2 + 7,\ 3 \cdot 3^2 + 7,\ 3 \cdot 5^2 + 7, \ldots\}$

Some special sets that have special names:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Z}^+$ = positive integers $= \{1, 2, 3, \ldots\}$
- $\mathbb{Q}$ = rational #'s $= \{p/q \mid p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \ne 0\}$
- $\mathbb{R}$ = real #'s
- $\mathbb{C}$ = Complex #'s

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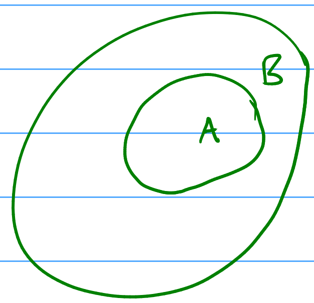
<u>Def'n</u> $A$ is a <u>subset</u> of $B$, written $A \subseteq B$, iff every element of $A$ is also an element of $B$. Formally,

$$\forall a : A.\ a \in B.$$




Question: Given a set $A$, is $A \subseteq A$?

According to the definition, yes! (Despite how we use the word in English.)

Aside: $A \subset B$ means "$A$ is a <u>proper subset</u> of $B$",
$(A \subseteq B) \wedge (A \ne B)$.

Defn Given sets $A$ and $B$, $A = B$ iff $(A \subseteq B) \wedge (B \subseteq A)$.

We actually use this to prove sets equal.

Ex. Let $A = \{2x+3 \mid x \in \mathbb{Z}\}$
$B = \{x \mid x \in \mathbb{Z},\ odd(x)\}$. Prove $A = B$.

Proof To show $A = B$, we will show $(A \subseteq B)$ and $(B \subseteq A)$.

- $(A \subseteq B)$. We must show $A \subseteq B$, that is, by definition, $\forall a : A.\ a \in B$. So let $a$ be an arbitrary element of $A$; we must show that $a \in B$.
  - If $a \in A$, then by definition of $A$, $a = 2x+3$ for some integer $x$. To show this is an element of $B$, we must show it is an odd integer.
    $a = 2x+3 = 2(x+1)+1$, hence $a$ is odd, so $a \in B$.

  Since $a$ was an arbitrary element of $A$, thus $\forall a : A.\ a \in B$. So by definition $A \subseteq B$.
- $(B \subseteq A)$ [omitted].

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Defn The empty set, written $\emptyset$ or $\{\}$, is the set with no elements.

Question. Is $\emptyset \subseteq \mathbb{N}$? YES!

By definition, this means $\forall x : \emptyset.\ x \in \mathbb{N}$.

$\forall$ with an empty domain !?!? T or F?

This is true. 3 arguments why:

① $\neg(\forall x : \emptyset.\ x \in \mathbb{N}) \equiv \exists x : \emptyset.\ x \notin \mathbb{N}$
which is clearly false, there are no elements in $\emptyset$.

② Think about whether someone has lied.

"Every February the 47th, I fly to the moon."
Silly, but true — you can't prove they are lying.

(3) $\forall x : \emptyset .\ x \in \mathbb{N} \equiv \forall x : \mathbb{N} .\ x \in \emptyset \to x \in \mathbb{N}.$

$$F \to ?$$

$$\equiv T.$$