Proof examples

Ex. Prove: if n is an odd integer then n^2 is also odd.

Step 1: translate to prop. logic -

$\forall n : \mathbb{Z}.\ Odd(n) \to Odd(n^2)$. ✓

equivalently, $\forall n : Odd.\ Odd(n^2)$.

$Odd(n) = \exists k : \mathbb{Z}.\ n = 2k+1$ or $Odd(n) = \neg Even(n)$

equivalent

easier

Proof. To show ~~$\forall n : \mathbb{Z}.\ Odd(n) \to Odd(n^2)$~~, let m be an arbitrary integer, and we will prove $Odd(m) \to Odd(m^2)$.

- To prove $Odd(m) \to Odd(m^2)$, suppose $Odd(m)$ is true, that is, there exists an integer k such that $m = 2k+1$. Then we must prove $Odd(m^2)$, that is, there exists an integer q such that $m^2 = 2q+1$.
  - m^2
    $=$ { substitute for $m = 2k+1$ }
    $(2k+1)^2$
    $=$ {algebra}
    $4k^2 + 4k + 1$
    $=$ {algebra}
    $2(2k^2 + 2k) + 1$

    Hence we can choose $q = 2k^2 + 2k$.
    Note this is an integer since k is.

  Therefore, since $Odd(m^2)$ is true if $Odd(m)$ is true, $Odd(m) \to Odd(m^2)$.

Therefore, since $Odd(m) \to Odd(m^2)$ for arbitrary

m. $\forall n : \mathbb{Z}.\ Odd(n) \rightarrow Odd(n^2)$.

This is how an experienced mathematician might write the same proof:

Proof: Let n be an odd integer, and suppose $n = 2k+1$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so n^2 is odd as well.

Example. Prove that if n is an integer and $3n+2$ is odd, then n is odd.

$\forall n : \mathbb{Z}.\ Odd(3n+2) \rightarrow Odd(n)$.

Proof. Let m be an arbitrary integer, then we must show $Odd(3m+2) \rightarrow Odd(m)$.

- So suppose $Odd(3m+2)$, that is, there is an integer k such that $3m+2 = 2k+1$. We must show $Odd(m)$, that is, $m = 2j+1$ for some integer j.
- Solving for m,

$3m + 2 = 2k + 1$

$\rightarrow$ {algebra}

$3m = 2k - 1$

$\rightarrow$ {algebra}

$m = \frac{2k-1}{3}$

Stuck! $\frac{2k-1}{3} = 2j+1$??

$\frac{2k-1}{3} = 2j + 1$

$\rightarrow \frac{2k-4}{3} = 2j$

$\rightarrow j = \frac{2k-4}{6} = \frac{k-2}{3}$ ← doesn't look like an integer!

FAIL :(

Attempt #2: prove the contrapositive!

Proof Let n be arbitrary integer, prove $Odd(3n+2) \to odd(n)$.

- we will prove the contraposition, that is,
  $\neg Odd(n) \to \neg odd(3n+2)$, that is,
  $Even(n) \to Even(3n+2)$.
  - Suppose $Even(n)$, that is, $n = 2k$ for some integer k. We will show $Even(3n+2)$, that is, $3n+2 = 2j$ for some integer j.

Assume $\neg odd \equiv Even$.

$$
\begin{aligned}
& 3n+2 \\
= \quad & \{\text{subst. } n = 2k\} \\
& 3(2k)+2 \\
= \quad & \{\text{alg.}\} \\
& 6k+2 \\
= \quad & \{\text{alg.}\} \\
& 2(3k+1)
\end{aligned}
$$

So $j = 3k+1$ works

Proof by contradiction

To prove p, prove $\neg p \to F$.

Example:

$Rational(n) = \exists p : \mathbb{Z}.\ \exists q : \mathbb{Z}.\ n = \frac{p}{q}$.

Prove: $\sqrt{2}$ is irrational, that is, $\neg Rational(\sqrt{2})$.

Proof. We will prove this by contradiction. That is, suppose $\sqrt{2}$ is rational; we will derive a contradiction.

- If $\sqrt{2}$ is rational, then by definition $\sqrt{2} = \frac{p}{q}$ for some integers p, q.

$\sqrt{2} = \frac{p}{q}$

$\rightarrow$ {Square both sides}

$2 = p^2/q^2$

$\rightarrow$ {mult. both sides by q^2}

$2q^2 = p^2$

$\rightarrow$ {definition, q^2 is an integer}

p^2 is even

$\rightarrow$ {contrapositive of theorem from above, $\forall n.\ \text{odd}(n) \rightarrow \text{odd}(n^2)$ }

$\rightarrow$

$\exists r : \mathbb{Z}.\ p = 2r.$

Now we can substitute $p = 2r$ into $2q^2 = p^2$

$2q^2 = p^2$

$\rightarrow$ {$p = 2r$}

$2q^2 = (2r)^2$

$\rightarrow$ {cancel 2 from both sides}

$q^2 = 2r^2$

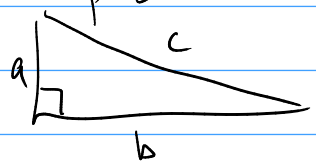
$\rightarrow$

q is also even.

If p, q are both divisible by 2, we can cancel it & then use the same argument again to cancel another factor of 2, & so on forever. — not possible.

Since supposing $\sqrt{2} = p/q$ led to a contradiction, therefore in fact it must be false — i.e. $\sqrt{2}$ is irrational.

Practice problem: in any right triangle



, $a + b > c$.