Logical Equivalences

Def'n if P and Q always have the same truth value no matter what, we say they are logically equivalent and write $P \equiv Q$.

Core equivalences / algebraic rules for logic.

Principles:

1. Most equivalences have a "opposite world" version where we switch T/F and $\wedge / \vee$.
2. We can get intuition by thinking in terms of addition & multiplication:
   - T is kind of like 1
   - F is kind of like 0
   - $\wedge$ is kind of like multiplication
   - $\vee$ is kind of like addition (w/ max value of 1).

Name	equivalence	"opposite world" version.
Identity	$p \wedge T \equiv p$	$p \vee F \equiv p$
Annihilation	$p \wedge F \equiv F$	$p \vee T \equiv T$
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
Commutativity	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$.
Associativity	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
	Contradiction: $p \wedge \neg p \equiv F$	Law of Excluded Middle: $p \vee \neg p \equiv T$

Double negation elimination: $\neg(\neg p) \equiv p$

Implication $p \rightarrow q \equiv \neg p \vee q$

Iff $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

Example. Prove $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

Strategy: use a series of equivalences

$$(p \to q) \wedge (p \to r) \equiv \dots \equiv \dots \equiv \dots \equiv p \to (q \wedge r).$$

relationship

$(p \to q) \wedge (p \to r)$

$\equiv$ { Implication } ← reason why relationship holds

$(\neg p \vee q) \wedge (\neg p \vee r)$

$\equiv$ { Distributivity }

$\neg p \vee (q \wedge r)$

$\equiv$ { Implication }

$p \to (q \wedge r)$

∎

Example. Show $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

$\neg(p \vee (\neg p \wedge q))$

$\equiv$ { De Morgan }

$\neg p \wedge \neg(\neg p \wedge q)$

$\equiv$ { De Morgan }

$\neg p \wedge (\neg(\neg p) \vee \neg q)$

$\equiv$ { 2x negation }

$\neg p \wedge (p \vee \neg q)$

$\equiv$ { Distributivity }

$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$

$\equiv$ { Contradiction }

$F \vee (\neg p \wedge \neg q)$

$\equiv$ { Identity }

$\neg p \wedge \neg q$

∎

$\equiv$ De Morgan

$\neg(p \vee q)$