

Discrete Math HW 5: Learning goals S_2, S_3, P_2 (solutions)

S_2 : I can evaluate and construct sets using union, intersection, difference, and complement of sets, and sets defined via set builder notation.

Exercise 1 Let A , B , and C represent the following sets:

$$A = \{5, 8, 13, 20\}$$

$$B = \{n \mid n \in \mathbb{N} \wedge n < 10\}$$

$$C = \{10k \mid k \in \mathbb{Z}\}$$

Write out the elements in the sets corresponding to each expression below.

(a) $A - B$

$$\{5, 8, 13, 20\} - \{n \mid n \in \mathbb{N} \wedge n < 10\} = \{13, 20\}$$

(b) $B \cap C$

$B \cap C$ consists of all the natural numbers which are less than ten and also a multiple of ten. There is only one such number, namely, $B \cap C = \{0\}$.

(c) $A \cap B \cap C$

Since we already know $B \cap C = \{0\}$, we are looking for $A \cap \{0\}$. But 0 is not an element of A , so in fact $A \cap B \cap C = \emptyset$ (the empty set).

(d) $\{x \mid (x \in C \cup A) \wedge (|x| \leq 10)\}$

We are looking for all the elements in either C or A which have an absolute value of at most 10: $\{-10, 0, 5, 8, 10\}$.

S_3 : I can list all the elements in a power set, Cartesian product, or disjoint union of sets, or count them without listing them all.

Exercise 2 List all the elements of $\mathcal{P}(\{3, 5, 7\})$.

$$\{\emptyset, \{3\}, \{5\}, \{7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{3, 5, 7\}\}.$$

Exercise 3 List all the elements of $\mathcal{P}(\{1, 2, 3, 4\})$.

Just for variety, let's make a table to organize things this time (this is not required but you are welcome to do it if you find it helpful).

1?	2?	3?	4?	subset
✓	✓	✓	✓	{1,2,3,4}
✓	✓	✓		{1,2,3}
✓	✓		✓	{1,2,4}
✓	✓			{1,2}
✓		✓	✓	{1,3,4}
✓		✓		{1,3}
✓			✓	{1,4}
✓				{1}
	✓	✓	✓	{2,3,4}
	✓	✓		{2,3}
	✓		✓	{2,4}
	✓			{2}
		✓	✓	{3,4}
		✓		{3}
			✓	{4}
				∅

Exercise 4 List all the elements of $\{1,2\} \times \{-3,-4,-5\}$.

There are six: $\{(1,-3), (1,-4), (1,-5), (2,-3), (2,-4), (2,-5)\}$.

Remember that the order that we list the pairs does not matter (but the order of the numbers within each pair *does* matter). For example,

$$\{(1,-4), (2,-4), (1,-5), (2,-3), (2,-5), (1,-3)\}$$

is also correct.

Exercise 5 List all the elements of $\{A\} \times \{B,C,D,E\}$.

There are four: $\{(A,B), (A,C), (A,D), (A,E)\}$.

Exercise 6 List all the elements of $\{A,B,C\} \times \{X,Y,Z\}$.

There are nine: $\{(A,X), (A,Y), (A,Z), (B,X), (B,Y), (B,Z), (C,X), (C,Y), (C,Z)\}$.

Exercise 7 Evaluate: $|\mathcal{P}(\{1,\dots,5\})|$.

In general, for a finite set A , $|\mathcal{P}(A)| = 2^{|A|}$. So $|\mathcal{P}(\{1,\dots,5\})| = 2^5 = 32$.

Exercise 8 Evaluate: $|\{A,B,C,\dots,Z\} \times \{1,2,3,\dots,10\}|$.

In general, for finite sets, $|A \times B| = |A| \times |B|$. So

$$|\{A,B,C,\dots,Z\} \times \{1,2,3,\dots,10\}| = |\{A,B,C,\dots,Z\}| \times |\{1,2,3,\dots,10\}| = 26 \times 10 = 260.$$



P2: I can reproduce proofs about sets, set operations, and the subset relation.

Do at least one of the following exercises.

Exercise 9 Prove that for all sets S and T ,

$$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

Proof. Let S and T be arbitrary sets. To show $\overline{S \cap T} = \overline{S} \cup \overline{T}$, we will show that each is a subset of the other.

To show $\overline{S \cap T} \subseteq \overline{S} \cup \overline{T}$, let $x \in \overline{S \cap T}$; we must show $x \in \overline{S} \cup \overline{T}$ as well.

$$\begin{aligned} & x \in \overline{S \cap T} \\ \equiv & \quad \quad \quad \{ \text{Definition of set complement} \} \\ & \neg(x \in S \cap T) \\ \equiv & \quad \quad \quad \{ \text{Definition of set intersection} \} \\ & \neg(x \in S \wedge x \in T) \\ \equiv & \quad \quad \quad \{ \text{De Morgan} \} \\ & \neg(x \in S) \vee \neg(x \in T) \\ \equiv & \quad \quad \quad \{ \text{Definition of set complement} \} \\ & (x \in \overline{S}) \vee (x \in \overline{T}) \\ \equiv & \quad \quad \quad \{ \text{Definition of set union} \} \\ & x \in \overline{S} \cup \overline{T} \end{aligned}$$

To show $\overline{S} \cup \overline{T} \subseteq \overline{S \cap T}$, we must show that any $x \in \overline{S} \cup \overline{T}$ is also an element of $\overline{S \cap T}$. But in fact we have already shown this: the previous sequence of equivalences works in reverse order, too.

□

Exercise 10 Prove that for all sets S , T , and U ,

$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U).$$

Proof. Let S , T , and U be arbitrary sets. We must show that both $S \cup (T \cap U) \subseteq (S \cup T) \cap (S \cup U)$ and vice versa.

Let x be an arbitrary element of $S \cup (T \cap U)$. We must show $x \in (S \cup T) \cap (S \cup U)$ as well.

$$\begin{aligned} & x \in S \cup (T \cap U) \\ \equiv & \quad \quad \quad \{ \text{Definition of set union} \} \end{aligned}$$



$$\begin{aligned}
& (x \in S) \vee (x \in T \cap U) \\
\equiv & \quad \quad \quad \{ \text{Definition of set intersection} \} \\
& (x \in S) \vee ((x \in T) \wedge (x \in U)) \\
\equiv & \quad \quad \quad \{ \vee \text{ distributes over } \wedge \} \\
& ((x \in S) \vee (x \in T)) \wedge ((x \in S) \vee (x \in U)) \\
\equiv & \quad \quad \quad \{ \text{Definition of set union} \} \\
& (x \in S \cup T) \wedge (x \in S \cup U) \\
\equiv & \quad \quad \quad \{ \text{Definition of set intersection} \} \\
& x \in (S \cup T) \cap (S \cup U)
\end{aligned}$$

Let x be an arbitrary element of $(S \cup T) \cap (S \cup U)$. Then we must show that $x \in S \cup (T \cap U)$. However, the sequence of equivalences shown above already does this in reverse.

□

