Discrete Math HW 7: Learning goals F3, P3 (solutions)

due Monday, March 16

F3: I can determine whether a given function is 1-1 and/or onto.

Exercise 1 For each of the following functions, determine whether the function is injective (1-1), surjective (onto), both, or neither, and justify your assertions. Feel free to use Disco to help explore the behavior of these functions, though you are not required to do so. For full credit, complete at least 4.

(a) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x - 1$

This function is a bijection (*i.e.* both 1-1 and onto).

- It is 1-1 since $(f(x) = f(y)) \to (x - 1 = y - 1) \to (x = y)$.
- It is onto since for any $y \in \mathbb{Z}$, $f(y + 1) = y$.

(b) $f : \mathbb{N} \to \mathbb{Z}; f(x) = x - 1$

- This function is 1-1 for the same reason as the previous function.
- However, it is not onto: for example, there is no natural number n such that $f(n) = -7$.

(c) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x^3$

- f is 1-1, since every integer has a unique cube.
- However, f is not onto. For example, there is no integer whose cube is 3.

(d) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; f(a, b) = a + b$

- f is not 1-1. For example, $(2, 3) \neq (1, 4)$, but f sends them both to the same output: $f(2, 3) = f(1, 4) = 5$.
- f is onto. For any $n \in \mathbb{N}$ we have $f(0, n) = n$.

(e) $f : \mathbb{Q} \to \mathbb{Q}; f(x) = 5x + 7$ f is a bijection.

- f is 1-1 since $5x + 7 = 5y + 7$ implies $x = y$.
- f is onto: for any rational number q, $(q - 7)/5$ is a rational number such that $f((q - 7)/5) = q$.

(f) $f : \mathbb{N} \to \mathbb{Z}; f(x) = x$

- f is 1-1: $f(x) = f(y)$ clearly implies $x = y$.
- However, f is not onto. For example, there is no natural number n such that $f(n) = -5$.

(g) $f : \mathbb{Z} \to \mathbb{N}$; $f(x) = |x|$

- f is not 1-1: for example, $f(-2) = f(2) = 2$.
- f is onto: for any natural number n, $f(n) = n$.

P3: I can reproduce proofs about 1-1, onto, and bijective functions.

Exercise 2 Prove: if $f : A \to B$ and $g : B \to C$ are both one-to-one, then the composite function $g \circ f : A \to C$, defined by $(g \circ f)(a) = g(f(a))$, is also one-to-one.

Proof. Suppose $f : A \to B$ and $g : B \to C$ are one-to-one functions. To show that $g \circ f$ is also one-to-one, we will prove

$$\forall x, y : A.\ ((g \circ f)(x) = (g \circ f)(y)) \to (x = y).$$

So let $x, y \in A$ be arbitrary, and suppose $(g \circ f)(x) = (g \circ f)(y)$. We have

$$
\begin{array}{ll}
\quad (g \circ f)(x) = (g \circ f)(y) & \\
\to & \{ \quad \text{Definition of } \circ \quad \} \\
\quad g(f(x)) = g(f(y)) & \\
\to & \{ \quad g \text{ is one-to-one} \quad \} \\
\quad f(x) = f(y) & \\
\to & \{ \quad f \text{ is one-to-one} \quad \} \\
\quad x = y &
\end{array}
$$

□

Exercise 3 Prove: if $f : A \to B$ and $g : B \to C$ are both onto, then the composite function $g \circ f : A \to C$, defined by $(g \circ f)(a) = g(f(a))$, is also onto.

Proof. Suppose $f : A \to B$ and $g : B \to C$ are onto functions. To show that $g \circ f$ is also onto, we will prove

$$\forall c : C.\ \exists a : A.\ (g \circ f)(c) = a.$$

So let $c \in C$ be arbitrary. First, since g is onto, there must exist $b \in B$ such that $g(b) = c$. But then since f is onto, there must exist $a \in A$ such that $f(a) = b$. Putting these together, we have $(g \circ f)(a) = g(f(a)) = g(b) = c$. □