Discrete Math Challenge HW 3 (2 points)

Recall from class that a rational number is defined as one which can be expressed as the ratio of two integers. That is,

$$\text{Rational}(x) \equiv \exists p\!:\!\mathbb{Z}.\ \exists q\!:\!\mathbb{Z}.\ (x = p/q).$$

(a) Prove that the sum of a rational number and an irrational number must be irrational.

(b) Prove that the product of a nonzero rational number and an irrational number must be irrational.

(c) Give an example showing that it is possible for the sum of two irrational numbers to be rational.

(d) Prove that $\sqrt{2} + \sqrt{3}$ is in fact irrational.

Hint: simplify $(\sqrt{2} + \sqrt{3})^2$ and use the results from the previous exercises to show that $(\sqrt{2} + \sqrt{3})^2$ must be irrational; then explain why this shows $\sqrt{2} + \sqrt{3}$ must be irrational as well. In class, we proved that $\sqrt{2}$ is irrational. You may assume that $\sqrt{3}$ is also irrational. In fact, $\sqrt{n}$ is always irrational whenever n is not a perfect square, though we won't be able to prove this until later in the course.