Discrete Math HW 13: Learning goals C1, C2 (solutions)

due anytime

C1: I can use the product, addition, and subtraction rules to count things.

Exercise 1 How many strings of eight uppercase English letters are there . . .

(a) if letters can be repeated?

There are 26 possibilities for each letter, and the choices are all independent, so the total is 26^8.

(b) if no letter can be repeated?

There are 26 choices for the first letter, 25 for the second, and so on, so the total is $26 \cdot 25 \cdot \cdots \cdot 19 = 26!/(26-8)!$.

(c) that start with X, if letters can be repeated?

There is only one choice for the first letter, and 26 for each of the other seven, so the total is 26^7.

(d) that start with X, if no letter can be repeated?

$26!/(26-7)!$

(e) that start and end with X, if letters can be repeated?

26^6

(f) that start with the letters BO (in that order), if letters can be repeated?

26^6

(g) that start and end with the letters BO (in that order), if letters can be repeated?

26^4

(h) that start or end with the letters BO (in that order), if letters can be repeated?

There are 26^6 strings that start with BO, 26^6 that end with BO, and 26^4 which both start and end with BO. Hence the number that start *or* end with BO is $26^6 + 26^6 - 26^4$.

Exercise 2 How many different functions are there from the set $\{A, B, C\}$ to *Bool*?

For each element of $\{A, B, C\}$ we have two choices of where to send it, for a total of $2^3 = 8$.

Exercise 3 How many bit strings are there of length 6 or less, not counting the empty string?

There are 2^k bit strings of length k, so the answer is $2^1 + 2^2 + \cdots + 2^6 = 2^7 - 2$.

C2: I can use the division rule and binomial coefficients to count things.

Exercise 4 How many ways are there to choose three weekdays on which to exercise?

This is just the number of ways to choose three things out of a set of seven, that is,

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

Exercise 5 How many ten-bit strings are there containing exactly three '1's?

This is the number of ways to choose the three positions out of ten which will contain the '1's:

$$\binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 4 = 120.$$

Exercise 6 How many ten-letter strings are there containing exactly three 'A's?

There are $\binom{10}{3}$ ways to choose the positions of the three As, and then 25 choices for each of the remaining 7 positions, for a total of

$$\binom{10}{3} \cdot 25^7.$$

Exercise 7 Consider the number of ways for 10 dogs and 6 cats to stand in a line so no two cats stand next to each other.

Hint: first position the dogs and then consider possible positions for the cats.

- First, how many ways are there if the dogs are all identical, and the cats are all identical?

  Once we put the dogs in a line, there are 11 positions where cats could stand (in between each pair of dogs and on the ends). Since cats can't stand next to each other, we just need to pick 6 of those 11 positions to put cats in, and there are $\binom{11}{6}$ ways to do this.

- What about if the dogs and cats are all distinct, so it matters what order they stand in?

  We can still pick positions for the cats in $\binom{11}{6}$ ways; but now there are 10! ways to put the dogs in a particular order, and 6! ways to put the cats in a particular order, so the total is

$$\binom{11}{6} \cdot 10! \cdot 6!.$$

Exercise 8 You are a mythical creature trainer! You have seven dragons and nine unicorns, and it's time to select a team to compete in the Annual Creatures of Myth Exposition (ACME). Note that a team is defined as a *set* of creatures, that is, the order of creatures on a team does not matter.

(a) How many ways are there to choose a team of five creatures if at least one dragon must be on the team?

Hint: How many possible teams are there? How many teams are there with no dragons?

There are $\binom{7+9}{5}$ possible teams (*i.e.* ways to choose a set of 5 creatures out of the total set of 16), and $\binom{9}{5}$ teams with no dragons (*i.e.* 5 chosen only from the set of 9 unicorns). Hence the number of teams with at least one dragon is just

$$\binom{7+9}{5} - \binom{9}{5} = 4242.$$

(b) How many ways are there to choose a team of five creatures if at least one dragon and at least one unicorn must be on the team?

We can subtract the teams with no dragons or no unicorns from the total:

$$\binom{7+9}{5} - \binom{9}{5} - \binom{7}{5} = 4221.$$