Discrete Math HW 7: Learning goals R4, P4

due Friday, April 11

P4: I can write the outline of a proof by (weak) induction or strong induction.

Exercise 1 Write the outline of a proof by induction for $\forall n : \mathbb{N}.\ P(n)$.

Exercise 2 Prove by induction: for all natural numbers n,

$$\sum_{1 \le j \le n+1} j \cdot 2^j = n \cdot 2^{n+2} + 2.$$

R4: I can come up with closed forms for recurrences and prove them via induction.

Exercise 3 For each recurrence below, list at least the first 5 terms of the sequence, and come up with a closed form. Then prove the closed form is correct, using either a proof by induction, or by showing that the closed form satisfies the recurrence when substituted for a_n.

(a) $a_n = a_{n-1} + 2; a_0 = 3$

(b) $a_n = 5a_{n-1}; a_0 = 1$

(c) $a_n = a_{n-1} + (2n+1); a_0 = 0$

(d) $a_n = 2na_{n-1}; a_0 = 3$

For full credit on this homework assignment, complete **either** Exercise 4 **or** Exercise 5 (or both, of course).

Exercise 4 This exercise concerns the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}.$$

(a) Write a Disco function `fracsum : N -> F` which computes the above sum for a given n. For example, `fracsum(2)` should output the sum $1/(1 \cdot 2) + 1/(2 \cdot 3)$.

Hint: your function should be recursive. Make sure your function is defined for *every* natural number input, including 0.

(b) Evaluate your function for some example inputs and look for a pattern. Make a conjecture of the form

$$\forall n : \mathbb{N}. \mathit{fracsum}(n) = \ldots.$$

(c) Use induction to prove your conjecture.

Exercise 5 This exercise concerns a variant of the *Ackermann function* (originally due to R. C. Buck), defined on pairs of natural number inputs as follows:

$$A(m,n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

(a) Find $A(1,0)$, $A(0,1)$, $A(1,1)$, and $A(2,2)$.

(b) Prove that $A(m,2) = 4$ for all natural numbers m.

(c) Prove that $A(1,n) = 2^n$ for all $n \geq 1$.

(d) Find each of the following values.

You might be able to use Disco to find some of them; for others, you can use the above theorems to help you calculate a result.

(i) $A(2,3)$

(ii) $A(3,3)$

(iii) (**optional**; +1 engagement point) $A(3,4)$
(*Hint:* feel free to leave your answer in a convenient algebraic form rather than expanding it out into its actual decimal digits.)