# *Discrete Math HW 7: Learning goals R4, P4 (solutions)*

*due Friday, April 11*

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*P4: I can write the outline of a proof by (weak) induction or strong induction.*

**Exercise 1** Write the outline of a proof by induction for $\forall n\!:\!\mathbb{N}.\ P(n)$.

*Proof.* We must prove $P(0)$, and $\forall k\!:\!\mathbb{N}.\ P(k) \to P(k+1)$.

> *Proof of* $P(0)$

> Let $k$ be an arbitrary natural number, and suppose $P(k)$. We must prove $P(k+1)$.
>
> > *Proof of* $P(k+1)$, *using* $P(k)$
>
> Therefore, since $k$ was arbitrary and we proved $P(k+1)$ while assuming $P(k)$, therefore $\forall k\!:\!\mathbb{N}.\ P(k) \to P(k+1)$.

Therefore, by induction, $P(n)$ is true for all natural numbers $n$. □

**Exercise 2** Prove by induction: for all natural numbers $n$,

$$\sum_{1 \le j \le n+1} j \cdot 2^j = n \cdot 2^{n+2} + 2.$$

*Proof.* Let $Q(n)$ be the proposition

$$Q(n) \equiv \left( \sum_{1 \le j \le n+1} j \cdot 2^j = n \cdot 2^{n+2} + 2 \right).$$

We will show $\forall n\!:\!\mathbb{N}.\ Q(n)$ by induction.

> The base case, $Q(0)$, says
>
> $$\sum_{1 \le j \le 1} j \cdot 2^j = 0 \cdot 2^{n+2} + 2.$$
>
> Both sides are equal to 2, so this is true.

Now let $k$ be an arbitrary natural number, and suppose $Q(k)$ is true, that is,

$$\sum_{1 \le j \le k+1} j \cdot 2^j = k \cdot 2^{k+2} + 2 \qquad (\text{KNOW}).$$

We will show $Q(k+1)$, that is,

$$\sum_{1 \le j \le k+2} j \cdot 2^j = (k+1) \cdot 2^{k+3} + 2 \qquad (\text{WANT}).$$

$$\sum_{1 \le j \le k+2} j \cdot 2^j$$

$=$ { pull out the $j = k+2$ term }

$$(k+2) \cdot 2^{k+2} + \sum_{1 \le j \le k+1} j \cdot 2^j$$

$=$ { assumption }

$$(k+2) \cdot 2^{k+2} + k \cdot 2^{k+2} + 2$$

$=$ { factor out $2^{k+2}$ }

$$((k+2)+k) \cdot 2^{k+2} + 2$$

$=$ { algebra }

$$(2k+2) \cdot 2^{k+2} + 2$$

$=$ { algebra }

$$(k+1) \cdot 2^{k+3} + 2$$

□

*R4: I can come up with closed forms for recurrences and prove them via induction.*

**Exercise 3** For each recurrence below, list at least the first 5 terms of the sequence, and come up with a closed form. Then prove the closed form is correct, using either a proof by induction, or by showing that the closed form satisfies the recurrence when substituted for $a_n$.

(a) $a_n = a_{n-1} + 2; a_0 = 3$

The first 5 terms are $3, 5, 7, 9, 11$. A closed form is $a_n = 2n + 3$. Let $P(n)$ be the proposition "$a_n = 2n + 3$"; we will prove by induction that $P(n)$ is true for all natural numbers $n$.

*Proof.* In the base case, $a_0 = 3$ by definitoin, and the closed form yields $2 \cdot 0 + 3 = 3$ also.

> Let $k$ be an arbitrary natural number, and suppose $P(k)$ is true, that is, $a_k = 2k+3$. We will show $P(k+1)$, that is, $a_{k+1} = 2(k+1)+3$.
>
> $a_{k+1}$
> $=$ { definition of the recurrence }
> $a_k + 2$
> $=$ { assumption }
> $2k+3+2$
> $=$ { algebra }
> $2(k+1)+3$
>
> Therefore, we have shown for an arbitrary $k$ that $P(k)$ implies $P(k+1)$.

Therefore, we have proved by induction that $P(n)$ is true for all natural numbers $n$, that is, $a_n = 2n+3$ is a correct closed form. □

(b) $a_n = 5a_{n-1}; a_0 = 1$

The first 5 terms are $1, 5, 25, 125, 625$, and a closed form is $a_n = 5^n$. We can prove this correct by substituting it into the recurrence:

- $5^0 = 1$, so this closed form satisfies the base case.
- Substituting $a_n = 5^n$ into both sides of $a_n = 5a_{n-1}$ yields $5^n = 5 \cdot 5^{n-1}$, which is true.

(c) $a_n = a_{n-1} + (2n+1); a_0 = 0$

The first 5 terms are $0, 3, 8, 15, 24$. To find a closed form we can either

- notice these terms are all one less than a perfect square, and hence conjecture that $a_n = (n+1)^2 - 1$, or
- we can calculate

$$\begin{aligned}\sum_{1 \le k \le n} (2k+1) &= 2\left(\sum_{1 \le k \le n} k\right) + \sum_{1 \le k \le n} 1 \\ &= 2\frac{n(n+1)}{2} + n \\ &= n^2 + 2n.\end{aligned}$$

These look different at first glance, but in fact $(n+1)^2 - 1 = (n^2+2n+1) - 1 = n^2 + 2n$.

If we calculated the answer, a proof by induction is not really needed, but it's still a good way to confirm that we didn't make a mistake. On the other hand, if we saw a pattern and guessed a formula, then we need a way to confirm our guess is correct.

We can show that the closed form $a_n = n^2 + 2n$ is correct by substituting it into the recurrence:

- For $n = 0$, $a_0 = 0$ by definition, and the closed form yields $0^2 + 2 \cdot 0 = 0$ also.
- Substituting the closed form into both sides of the recurrence yields

$$n^2 + 2n = (n-1)^2 + 2(n-1) + (2n+1);$$

  expanding out the right-hand side and collecting like terms, we find that

$$\begin{aligned}(n-1)^2 + 2(n-1) + (2n+1) &= n^2 - 2n + 1 + 2n - 2 + 2n + 1 \\ &= n^2 + 2n\end{aligned}$$

  so the two sides are equal.

(d) $a_n = 2na_{n-1}; a_0 = 3$

The first 5 terms are $3, 6, 24, 144, 1152$. We can find a closed form by expanding out the recurrence:

$$\begin{aligned}a_n &= 2na_{n-1} \\ &= 2n(2(n-1)a_{n-2}) = 2^2 \cdot n \cdot (n-1) \cdot a_{n-2} \\ &= 2^3 \cdot n \cdot (n-1) \cdot (n-2) \cdot a_{n-3} \\ &= \dots \\ &= 2^n \cdot n! \cdot a_0\end{aligned}$$

Hence a closed form is $a_n = 3 \cdot 2^n \cdot n!$, which we can show by substituting into the recurrence.

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For full credit on this homework assignment, complete **either** Exercise 4 **or** Exercise 5 (or both, of course).

**Exercise 4** This exercise concerns the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}.$$

(a) Write a Disco function `fracsum : N -> F` which computes the above sum for a given $n$. For example, `fracsum(2)` should output the sum $1/(1 \cdot 2) + 1/(2 \cdot 3)$.

*Hint*: your function should be recursive. Make sure your function is defined for *every* natural number input, including 0.

```
fracsum : N -> F
fracsum(0) = 0
fracsum(n) = fracsum(n .- 1) + 1/(n (n+1))
```

(b) Evaluate your function for some example inputs and look for a pattern. Make a conjecture of the form

$$\forall n : \mathbb{N}. \mathit{fracsum}(n) = \dots.$$

```
fracsum(0) = 0
fracsum(1) = 1/2
fracsum(2) = 2/3
fracsum(3) = 3/4
```

We conjecture that $\mathit{fracsum}(n) = n/(n+1)$ for all natural numbers $n$.

(c) Use induction to prove your conjecture.

*Proof.* Let $F(n)$ be the proposition $\mathit{fracsum}(n) = n/(n+1)$. We will prove $\forall n{:}\mathbb{N}.\ F(n)$ by induction.

> In the base case, $F(0)$ says $\mathit{fracsum}(0) = 0/1$, which is true, since $\mathit{fracsum}(0) = 0$ by definition.

> Now let $k$ be an arbitrary natural number, and suppose $F(k)$, that is, $\mathit{fracsum}(k) = k/(k+1)$. We must show $F(k+1)$, that is, $\mathit{fracsum}(k+1) = (k+1)/(k+2)$.
>
> > $\mathit{fracsum}(k+1)$
> > $= \{$ definition $\}$
> > $\mathit{fracsum}(k) + \frac{1}{(k+1)(k+2)}$
> > $= \{$ assumption (induction hypothesis) $\}$
> > $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$
> > $= \{$ algebra $\}$
> > $\frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$
> > $= \{$ algebra $\}$
> > $\frac{k^2+2k+1}{(k+1)(k+2)}$
> > $= \{$ algebra $\}$
> > $\frac{(k+1)^2}{(k+1)(k+2)}$
> > $= \{$ algebra $\}$
> > $\frac{k+1}{k+2}$

□

**Exercise 5** This exercise concerns a variant of the *Ackermann function* (originally due to R. C. Buck), defined on pairs of natural number inputs as follows:

$$A(m,n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

(a) Find $A(1,0)$, $A(0,1)$, $A(1,1)$, and $A(2,2)$.

- $A(1,0) = 0$
- $A(0,1) = 2$
- $A(1,1) = 2$
- $A(2,2) = A(1, A(2,1)) = A(1,2) = A(0, A(1,1)) = A(0,2) = 4$

(b) Prove that $A(m,2) = 4$ for all natural numbers $m$.

*Proof.* The proof is by induction.

When $m = 0$, $A(0,2) = 2 \cdot 2 = 4$ by definition.

Now let $k$ be an arbitrary natural number, and suppose $A(k,2) = 4$. We will show that $A(k+1,2) = 4$ as well.

$A(k+1,2)$
$=$ { definition of $A$ }
$A(k, A(k,1))$
$=$ { definition of $A$ }
$A(k,2)$
$=$ { assumption }
$4$

Therefore, by induction, $A(m,2) = 4$ for all $m$. □

(c) Prove that $A(1,n) = 2^n$ for all $n \geq 1$.

*Proof.* We will prove this by induction.

In the base case, when $n = 1$, we have $A(1,1) = 2$ by definition, and $2 = 2^1$.

> Now let $k \geq 1$ be an arbitrary natural number other than 0, and suppose the proposition is true for $k$, that is, $A(1,k) = 2^k$. Then we must show it is also true for $k+1$.
>
> > $A(1, k+1)$
> > $=$ { by definition ($k+1 \geq 2$ since $k \geq 1$) }
> > $A(0, A(1,k))$
> > $=$ { assumption }
> > $A(0, 2^k)$
> > $=$ { definition of $A$ }
> > $2 \cdot 2^k$
> > $=$ { algebra }
> > $2^{k+1}$

□

(d) Find each of the following values.

You might be able to use Disco to find some of them; for others, you can use the above theorems to help you calculate a result.

(i) $A(2,3)$

$A(2,3)$
$=$ { definition }
$A(1, A(2,2))$
$=$ { part (b) }
$A(1,4)$
$=$ { part (c) }
$16$

(ii) $A(3,3)$

$A(3,3)$
$=$ { definition }
$A(2, A(3,2))$
$=$ { part (b) }
$A(2,4)$
$=$ { definition }
$A(1, A(2,3))$
$=$ { part (d)(i) }
$A(1,16)$
$=$ { part (c) }
$2^{16}$

(iii) $A(3,4)$ (*Hint:* feel free to leave your answer in a convenient algebraic form rather than expanding it out into its actual decimal digits.)

First, let $2 \uparrow\uparrow n$ denote iterated exponentiation of $n$ copies of 2, that is, $2 \uparrow\uparrow 2 = 2^2$, $2 \uparrow\uparrow 3 = 2^{2^2}$, $2 \uparrow\uparrow 4 = 2^{2^{2^2}}$, and so on. We will show that $A(2,n) = 2 \uparrow\uparrow n$ for all $n \geq 1$. In the base case, $A(2,1) = 2$ by definition. Then, if we suppose $A(2,k) = 2 \uparrow\uparrow k$, then

$$
\begin{aligned}
& A(2,k+1) \\
= \quad & \{ \text{ definition } \} \\
& A(1, A(2,k)) \\
= \quad & \{ \text{ assumption } \} \\
& A(1, 2 \uparrow\uparrow k) \\
= \quad & \{ \text{ part (c) } \} \\
& 2^{2 \uparrow\uparrow k} \\
= \quad & \{ \text{ definition of } \uparrow\uparrow \} \\
& 2 \uparrow\uparrow (k+1)
\end{aligned}
$$

Now we may compute

$$
\begin{aligned}
& A(3,4) \\
= \quad & \{ \text{ definition } \} \\
& A(2, A(3,3)) \\
= \quad & \{ \text{ part (d)(ii) } \} \\
& A(2, 2^{16}) \\
= \quad & \{ \text{ previous lemma } \} \\
& 2 \uparrow\uparrow 2^{16},
\end{aligned}
$$

a truly gigantic number consisting of $65,536$ copies of 2 stacked in a tower of exponents.