Discrete Math HW 6: Learning goals F3, R1, R3 (solutions)

due Friday, April 4

F3: I can determine whether a given function is 1-1 and/or onto.

Exercise 1 For each of the following functions, determine whether the function is injective (1-1), surjective (onto), both, or neither, and justify your assertions. Feel free to use Disco to help explore the behavior of these functions, though you are not required to do so. For full credit, complete at least 4.

(a) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x - 1$

This function is a bijection (*i.e.* both 1-1 and onto).

- It is 1-1 since $(f(x) = f(y)) \to (x - 1 = y - 1) \to (x = y)$.
- It is onto since for any $y \in \mathbb{Z}$, $f(y + 1) = y$.

(b) $f : \mathbb{N} \to \mathbb{Z}; f(x) = x - 1$

- This function is 1-1 for the same reason as the previous function.
- However, it is not onto: for example, there is no natural number n such that $f(n) = -7$.

(c) $f : \mathbb{Z} \to \mathbb{Z}; f(x) = x^3$

- f is 1-1, since every integer has a unique cube.
- However, f is not onto. For example, there is no integer whose cube is 3.

(d) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; f(a, b) = a + b$

- f is not 1-1. For example, $(2, 3) \neq (1, 4)$, but f sends them both to the same output: $f(2, 3) = f(1, 4) = 5$.
- f is onto. For any $n \in \mathbb{N}$ we have $f(0, n) = n$.

(e) $f : \mathbb{Q} \to \mathbb{Q}; f(x) = 5x + 7$ f is a bijection.

- f is 1-1 since $5x + 7 = 5y + 7$ implies $x = y$.
- f is onto: for any rational number q, $(q - 7)/5$ is a rational number such that $f((q - 7)/5) = q$.

(f) $f : \mathbb{N} \to \mathbb{Z}; f(x) = x$

- f is 1-1: $f(x) = f(y)$ clearly implies $x = y$.
- However, f is not onto. For example, there is no natural number n such that $f(n) = -5$.

(g) $f : \mathbb{Z} \to \mathbb{N}$; $f(x) = |x|$

- f is not 1-1: for example, $f(-2) = f(2) = 2$.
- f is onto: for any natural number n, $f(n) = n$.

R1: I can compute the terms of a sequence defined via a recurrence.

Exercise 2 List the first five terms of each sequence defined below.

(a)

$$\begin{aligned} a_0 &= 1 \\ a_n &= a_{n-1} + 3 \quad (n \geq 1) \end{aligned}$$

$$1, 4, 7, 10, 13, \ldots$$

(b)

$$\begin{aligned} a_2 &= 7 \\ a_n &= a_{n-1} - 1 \quad (n \geq 3) \end{aligned}$$

$$7, 6, 5, 4, 3, \ldots$$

(c)

$$\begin{aligned} p_0 &= 0 \\ p_n &= 3p_{n-1} - 1 \quad (n \geq 1) \end{aligned}$$

$$0, -1, -4, -13, -40, \ldots$$

(d)

$$\begin{aligned} p_0 &= 0 \\ p_n &= p_{n-1} + n^2 \quad (n \geq 1) \end{aligned}$$

$$0, 1, 5, 14, 30, \ldots$$

(e)

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 1 \\ p_n &= 2p_{n-1} + p_{n-2} \quad (n \geq 2) \end{aligned}$$

$$1, 1, 3, 7, 17 \ldots$$

R3: I can evaluate sums involving arithmetic and geometric series.

Exercise 3 Evaluate each of the following sums. Show your work.
For credit, complete at least 4.

(a)

$$\begin{aligned}\sum_{0\le i\le 10}(2i-3) &= 2\left(\sum_{0\le i\le 10} i\right) - \sum_{0\le i\le 10} 3\\ &= 2\frac{10(10+1)}{2} - 11\cdot 3\\ &= 110-33 = 77\end{aligned}$$

(b)

$$\begin{aligned}\sum_{1\le k\le 20}(5-k) &= \left(\sum_{1\le k\le 20} 5\right) - \left(\sum_{1\le k\le 20} k\right)\\ &= 20\cdot 5 - \frac{20\cdot 21}{2}\\ &= 100-210 = -110\end{aligned}$$

(c)

$$\begin{aligned}\sum_{0\le j\le 50}(3j-7) &= 3\left(\sum_{0\le j\le 50} j\right) - \left(\sum_{0\le j\le 50} 7\right)\\ &= 3\frac{50\cdot 51}{2} - 51\cdot 7\\ &= 3468\end{aligned}$$

(d)

$$\begin{aligned}&7+10+13+16+19+\cdots+100\\ &= \sum_{0\le k\le 31} 3k+7\\ &= 3\left(\sum_{0\le k\le 31} k\right) + \left(\sum_{0\le k\le 31} 7\right)\\ &= 3\frac{31\cdot 32}{2} + 32\cdot 7\\ &= 1712\end{aligned}$$

(e)

$$\begin{aligned}\sum_{7\le i\le 29} i &= \left(\sum_{1\le i\le 29} i\right) - \left(\sum_{1\le i\le 6} i\right)\\ &= \frac{29\cdot 30}{2} - \frac{6\cdot 7}{2}\\ &= 414\end{aligned}$$

(f)

$$\begin{aligned}\sum_{0 \le k \le 10} (2^k - 1) &= \left(\sum_{0 \le k \le 10} 2^k \right) - \left(\sum_{0 \le k \le 10} 1 \right) \\ &= \frac{2^{11} - 1}{2 - 1} - 11 \\ &= 2047 - 11 = 2036\end{aligned}$$

(g)

$$\begin{aligned}&1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^7} \\ &= \sum_{0 \le k \le 7} \left(\frac{1}{3} \right)^k \\ &= \frac{1 - (1/3)^8}{1 - 1/3} \\ &= 3280/2187\end{aligned}$$