# *Discrete Math HW 5: Learning goal F1 (solutions)*

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*F1: I can determine whether a relation is reflexive, transitive, symmetric, antisymmetric, or an equivalence, and give examples of relations with these properties.*

**Exercise 1** For each relation below, state whether it is reflexive, transitive, symmetric, and/or antisymmetric. Give brief reasoning/justification/proof for each.

For credit, do at least 6.

1. The $<$ relation on $\mathbb{Z}$, that is, $\{(a, b) \mid a \in Z, b \in Z, a < b\}$
   - Not reflexive; it is never true that $a < a$
   - Transitive: if $a < b$ and $b < c$, then $a < c$
   - Not symmetric: if $a < b$ then it is not true that $b < a$
   - Technically, it is antisymmetric: since it is impossible to have both $a < b$ and $b < a$, it is true that $(a < b) \wedge (b < a) \rightarrow (a = b)$ (because "false implies anything is true").
2. The $\leq$ relation on $\mathbb{Z}$, that is, $\{(a, b) \mid a \in Z, b \in Z, a \leq b\}$
   - Reflexive: for all $a \in \mathbb{Z}$, we have $a \leq a$.
   - Transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$.
   - Not symmetric: for example, $1 \leq 2$ but it is not true that $2 \leq 1$.
   - Antisymmetric: if $a \leq b$ and $b \leq a$ then it must be the case that $a = b$.
3. $\{(x, y) \mid (\text{Odd}(x) \wedge \text{Odd}(y)) \vee (\text{Even}(x) \wedge \text{Even}(y))\}$
   - Reflexive: $(x, x)$ is in the relation for any integer $x$, since if $x$ is odd then $(\text{Odd}(x) \wedge \text{Odd}(x))$ is true, or similarly if $x$ is even.
   - Transitive: if $(x, y)$ are related and $(y, z)$ are related, then either all three are odd, or all three are even. In either case, $(x, z)$ must be related as well.
   - Symmetric: if $(x, y)$ are related then they are both even or both odd, so $(y, x)$ are related as well.
   - Not antisymmetric: for example $(2, 4)$ and $(4, 2)$ are in the relation, but it is not true that $2 = 4$.
4. $\{(x, y) \mid (\text{Odd}(x) \wedge \text{Even}(y)) \vee (\text{Even}(x) \wedge \text{Odd}(y))\}$

- Not reflexive: for example, 1 is not related to itself because it cannot be both odd and even.
- Not transitive: for example, $(1,2)$ is in the relation, and so is $(2,3)$, but $(1,3)$ is not.
- Symmetric: whenever $(a,b)$ is in the relation, $(b,a)$ must be as well, since one is even and one is odd.
- Not antisymmetric: for example, $(1,2)$ and $(2,1)$ are related but $1 \neq 2$.

5. $\{(a,b) \mid a \land b \equiv \mathit{True}\}$

   - It is not reflexive. False (or any false proposition) is not related to itself.
   - Transitive: the only way for $(a,b)$ to be related is if $a$ and $b$ are both separately equivalent to True. In that case if $(a,b)$ are related and $(b,c)$ are related then $a \equiv b \equiv c \equiv T$ so $(a,c)$ are related.
   - Symmetric: since $\land$ is symmetric, so is this relation.
   - Whether it is antisymmetric or not depends on whether we take this to be a relation over the set of Booleans, or over the set of all propositions. If the latter, it is not antisymmetric: just because $a \land b \equiv b \land a \equiv \mathit{True}$ does not mean $a$ and $b$ are exactly the same proposition. On the other hand, if we take the relation to be over the Booleans rather than all propositions, then it is antisymmetric, since $(T,T)$ is the only pair in the relation at all.

6. $\{(x,x) \mid x \in \mathbb{N}, \frac{x}{x} = 1\}$

   This is almost just the equality relation over the natural numbers, *except* that 0 is not related to itself ($0/0$ is not equal to 1; it is undefined). Hence it is transitive, symmetric, and antisymmetric, but NOT reflexive!

7. $\{((a,b),(c,d)) \mid a,b,c,d \in \mathbb{N}, a+d = b+c\}$

   - Reflexive: any pair $(a,b)$ is related to itself, since $a+b = b+a$.
   - Transitive: let $(a,b)$, $(c,d)$, and $(e,f)$ be arbitrary pairs of natural numbers, and suppose that $(a,b)$ is related to $(c,d)$, and $(c,d)$ is related to $(e,f)$. By definition, that means

     $$a + d = b + c$$

     and

     $$c + f = d + e.$$

     Adding these two equations yields

     $$a + d + c + f = b + c + d + e$$

but we can cancel $c + d$ from both sides, leaving us with

$$a + f = b + e,$$

which by definition shows that $(a, b)$ is related to $(e, f)$.

- Symmetric: suppose $(a, b)$ and $(c, d)$ are related, that is, $a + d = b + c$. In order for $(c, d)$ and $(a, b)$ to be related, by definition we would need $c + b = d + a$, but that is the same as $a + d = b + c$ since addition is commutative and equality is symmetric.
- It is not antisymmetric. For example, $(2, 4)$ is related to $(3, 5)$ and also vice versa, but these pairs of numbers are not equal.

As an interesting aside, this is a formal way we could define the integers starting from just natural numbers: integers can be defined as equivalence classes of pairs of natural numbers under this equivalence relation!

8. $\{((p, q), (r, s)) \mid p, q, r, s \in \mathbb{Z}, ps = qr\}$

   The analysis of this relation is very similar to the previous one: it is reflexive, transitive, and symmetric (*i.e.* an equivalence relation), and the proofs are very similar. Just like the previous relation can be used to define the integers starting from only natural numbers, this relation can be used to define rational numbers starting from only integers.

9. The empty relation, $\varnothing$
   - Not reflexive, since nothing is related to itself.
   - But it is transitive, symmetric, and antisymmetric, all for the "trivial" reason that false implies anything is true.

10. The complete relation on $\mathbb{Z}$, $\{(a, b) \mid a, b \in \mathbb{Z}\}$
    - Since everything is related to everything, it is reflexive, transitive, and symmetric.
    - However, it is not antisymmetric, since, for example, $(2, 7)$ and $(7, 2)$ are in the relation but $2 \neq 7$.

**Exercise 2** Give an example of a relation with each given set of properties.

1. Transitive but not reflexive: $<$
2. Reflexive and transitive but not symmetric: $\leq$
3. An equivalence relation: $=$