Discrete Math HW 4: Learning goals S1–S3, P2 (solutions)

S1: I can state the definitions, and determine membership, of standard sets such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Exercise 1 Name three examples of numbers which are members of $\mathbb{Z}$ but not $\mathbb{N}$.

$\mathbb{N}$ includes zero and all positive whole numbers; $\mathbb{Z}$ includes negative numbers as well. Therefore, -3, -7, and -49 are three examples of numbers in $\mathbb{Z}$ but not $\mathbb{N}$.

Exercise 2 Name three examples of numbers which are in $\mathbb{Q}$ but not $\mathbb{Z}$.

$\mathbb{Q}$ includes all positive and negative fractions. Three examples of elements of $\mathbb{Q}$ which are not in $\mathbb{Z}$ are $1/2$, $19/47$, and $-2/3$.

S2: I can evaluate and construct sets using union, intersection, difference, and complement of sets, and sets defined via set comprehension notation.

Exercise 3 Let A, B, and C represent the following sets:

$$\begin{aligned} A &= \{5, 8, 13, 20\} \\ B &= \{n \mid n \in \mathbb{N} \land n < 10\} \\ C &= \{10k \mid k \in \mathbb{Z}\} \end{aligned}$$

Write out the elements in the sets corresponding to each expression below.

(a) $A - B$

$\{5, 8, 13, 20\} - \{n \mid n \in \mathbb{N} \land n < 10\} = \{13, 20\}$

(b) $B \cap C$

$B \cap C$ consists of all the natural numbers which are less than ten and also a multiple of ten. There is only one such number, namely, $B \cap C = \{0\}$.

(c) $A \cap B \cap C$

Since we already know $B \cap C = \{0\}$, we are looking for $A \cap \{0\}$. But 0 is not an element of A, so in fact $A \cap B \cap C = \varnothing$ (the empty set).

(d) $\{x \mid (x \in C \cup A) \wedge (|x| \leq 10)\}$

We are looking for all the elements in either C or A which have an absolute value of at most 10: $\{-10, 0, 5, 8, 10\}$.

S3: I can list all the elements in a power set, Cartesian product, or disjoint union of sets, or count them without listing them all.

Exercise 4 List all the elements of $\mathcal{P}(\{3,5,7\})$.

$\{\varnothing, \{3\}, \{5\}, \{7\}, \{3,5\}, \{3,7\}, \{5,7\}, \{3,5,7\}\}$.

Exercise 5 List all the elements of $\mathcal{P}(\{1,2,3,4\})$.

Just for variety, let's make a table to organize things this time.

1?	2?	3?	4?	subset
✓	✓	✓	✓	$\{1,2,3,4\}$
✓	✓	✓		$\{1,2,3\}$
✓	✓		✓	$\{1,2,4\}$
✓	✓			$\{1,2\}$
✓		✓	✓	$\{1,3,4\}$
✓		✓		$\{1,3\}$
✓			✓	$\{1,4\}$
✓				$\{1\}$
	✓	✓	✓	$\{2,3,4\}$
	✓	✓		$\{2,3\}$
	✓		✓	$\{2,4\}$
	✓			$\{2\}$
		✓	✓	$\{3,4\}$
		✓		$\{3\}$
			✓	$\{4\}$
				$\varnothing$

Exercise 6 List all the elements of $\{1,2\} \times \{-3,-4,-5\}$.

There are six: $\{(1,-3),(1,-4),(1,-5),(2,-3),(2,-4),(2,-5)\}$. Remember that the order that we list the pairs does not matter (but the order of the numbers within each pair *does* matter). For example,

$$\{(1,-4),(2,-4),(1,-5),(2,-3),(2,-5),(1,-3)\}$$

is also correct.

Exercise 7 List all the elements of $\{A\} \times \{B,C,D,E\}$.

There are four: $\{(A,B),(A,C),(A,D),(A,E)\}$.

Exercise 8 List all the elements of $\{A, B, C\} \times \{X, Y, Z\}$.

There are nine: $\{(A, X), (A, Y), (A, Z), (B, X), (B, Y), (B, Z), (C, X), (C, Y), (C, Z)\}$.

Exercise 9 Evaluate: $|\mathcal{P}(\{1, \ldots, 5\})|$.

In general, for a finite set A, $|\mathcal{P}(A)| = 2^{|A|}$. So $|\mathcal{P}(\{1, \ldots, 5\})| = 2^5 = 32$.

Exercise 10 Evaluate: $|\{A, B, C, \ldots, Z\} \times \{1, 2, 3, \ldots, 10\}|$.

In general, for finite sets, $|A \times B| = |A| \times |B|$. So

$$|\{A, B, C, \ldots, Z\} \times \{1, 2, 3, \ldots, 10\}| = |\{A, B, C, \ldots, Z\}| \times |\{1, 2, 3, \ldots, 10\}| = 26 \times 10 = 260.$$

P2: I can write proofs about sets, set operations, and the subset relation.

Do at least one of the following exercises.

Exercise 11 Prove that for all sets S and T,

$$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

Proof. Let S and T be arbitrary sets. To show $\overline{S \cap T} = \overline{S} \cup \overline{T}$, we will show that each is a subset of the other.

> To show $\overline{S \cap T} \subseteq \overline{S} \cup \overline{T}$, let $x \in \overline{S \cap T}$; we must show $x \in \overline{S} \cup \overline{T}$ as well.
>
> $$\begin{aligned}
> & x \in \overline{S \cap T} \\
> \equiv \quad & \{ \text{ Definition of set complement } \} \\
> & \neg(x \in S \cap T) \\
> \equiv \quad & \{ \text{ Definition of set intersection } \} \\
> & \neg(x \in S \land x \in T) \\
> \equiv \quad & \{ \text{ De Morgan } \} \\
> & \neg(x \in S) \lor \neg(x \in T) \\
> \equiv \quad & \{ \text{ Definition of set complement } \} \\
> & (x \in \overline{S}) \lor (x \in \overline{T}) \\
> \equiv \quad & \{ \text{ Definition of set union } \} \\
> & x \in \overline{S} \cup \overline{T}
> \end{aligned}$$

> To show $\overline{S} \cup \overline{T} \subseteq \overline{S \cap T}$, we must show that any $x \in \overline{S} \cup \overline{T}$ is also an element of $\overline{S \cap T}$. But in fact we have already shown this: the previous sequence of equivalences works in reverse order, too.

□

Exercise 12 Prove that for all sets S, T, and U,

$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U).$$

Proof. Let S, T, and U be arbitrary sets. We must show that both $S \cup (T \cap U) \subseteq (S \cup T) \cap (S \cup U)$ and vice versa.

Let x be an arbitrary element of $S \cup (T \cap U)$. We must show $x \in (S \cup T) \cap (S \cup U)$ as well.

$$
\begin{aligned}
& x \in S \cup (T \cap U) \\
\equiv & \qquad \{ \text{ Definition of set union } \} \\
& (x \in S) \vee (x \in T \cap U) \\
\equiv & \qquad \{ \text{ Definition of set intersection } \} \\
& (x \in S) \vee ((x \in T) \wedge (x \in U)) \\
\equiv & \qquad \{ \vee \text{ distributes over } \wedge \} \\
& ((x \in S) \vee (x \in T)) \wedge ((x \in S) \vee (x \in U)) \\
\equiv & \qquad \{ \text{ Definition of set union } \} \\
& (x \in S \cup T) \wedge (x \in S \cup U) \\
\equiv & \qquad \{ \text{ Definition of set intersection } \} \\
& x \in (S \cup T) \cap (S \cup U)
\end{aligned}
$$

Let x be an arbitrary element of $(S \cup T) \cap (S \cup U)$. Then we must show that $x \in S \cup (T \cap U)$. However, the sequence of equivalences shown above already does this in reverse.

□