Discrete Math HW 3: Learning goal P1 (solutions)

P1: I can write an appropriate proof outline for a given propositional logic formula.

Exercise 1 Write an outline for a proof of $P \to (Q \land R)$.

Proof. We must show $P \to (Q \land R)$. So suppose P; we must show $Q \land R$.

> In order to show $Q \land R$, we will show both separately.
>
> > *Proof of Q (using P)*
>
> > *Proof of R (using P)*
>
> Since we have shown Q and R separately, therefore we have proved $Q \land R$.

We showed $Q \land R$ under the supposition that P is true; therefore $P \to (Q \land R)$. □

Exercise 2 Write an outline for a proof of $P \leftrightarrow \neg Q$.

Proof. To show $P \leftrightarrow \neg Q$, we will show both directions of the implication.

> ($\to$) First, we will show $P \to \neg Q$. Suppose P; we must show $\neg Q$.
>
> > To show $\neg Q$, we will prove $Q \to F$. So suppose Q is true; we will derive a contradiction.
> >
> > > *Proof that Q (and P) together make a contradiction.*
> >
> > Since supposing Q leads to a contradiction, therefore $\neg Q$.
>
> We proved $\neg Q$ under the supposition P, so $P \to \neg Q$.

> ($\leftarrow$) Next, we will show $\neg Q \to P$. Suppose $\neg Q$; we will show P.

> > *Proof of P (using ¬Q)*
>
> Therefore, $\neg Q \to P$.

We have shown both $P \to \neg Q$ and $\neg Q \to P$. Therefore, $P \leftrightarrow \neg Q$. □

Exercise 3 Write an outline for a proof of $\forall x : D.\ P(x) \wedge Q(x)$.

Proof. Let d be an arbitrary element of D. We will show $P(d) \wedge Q(d)$.

> To prove $P(d) \wedge Q(d)$, we will prove both separately.
>
> > *Proof of* $P(d)$
>
> > *Proof of* $Q(d)$

Therefore, since we proved $P(d) \wedge Q(d)$ for an arbitrary element of D, in fact it holds for all elements of D, that is, $\forall x : D.\ P(x) \wedge Q(x)$. □

Exercise 4 Write an outline for a proof of $\exists n : D.\ P(n) \to Q(n)$. Use a proof by contrapositive for the implication.

Proof. To show $\exists n : D.\ P(n) \to Q(n)$, we will show that $P(d) \to Q(d)$ holds for the specific value d in the domain D.

> We will show $P(d) \to Q(d)$ using the contrapositive, $\neg Q(d) \to \neg P(d)$. So suppose $\neg Q(d)$; we will show $\neg P(d)$.
>
> > *Proof of* $\neg Q(d)$ *(using* $\neg P(d)$*)*
>
> Therefore $\neg Q(d) \to \neg P(d)$ since we proved $\neg P(d)$ under the supposition that $\neg Q(d)$ is true.

Since $P(d) \to Q(d)$ holds for the specific value d, we have proved that such an element exists, that is, $\exists x : D.\ P(x) \to Q(x)$. □

Exercise 5 Prove: for all integers m and n, if mn is even, then either m is even or n is even (or both).

Translating to propositional logic, we are asked to prove

$$\forall m : \mathbb{Z}.\ \forall n : \mathbb{Z}.\ \text{Even}(mn) \to (\text{Even}(m) \vee \text{Even}(n)).$$

Proof. Let a and b be arbitrary integers; we will prove that $\text{Even}(ab) \to (\text{Even}(a) \vee \text{Even}(b))$.

> To prove this implication, we will prove the contrapositive, that is,
>
> $$\neg(\text{Even}(a) \lor \text{Even}(b)) \to \neg\text{Even}(ab)$$
> $$\equiv$$
> $$(\neg\text{Even}(a) \land \neg\text{Even}(b)) \to \neg\text{Even}(ab)$$
>
> Since we are assuming $\neg\text{Even} \equiv \text{Odd}$, this is equivalent to showing
>
> $$(\text{Odd}(a) \land \text{Odd}(b)) \to \text{Odd}(ab).$$
>
> > So suppose $\text{Odd}(a) \land \text{Odd}(b)$; we will show $\text{Odd}(ab)$.
> >
> > > Since we are supposing $\text{Odd}(a) \land \text{Odd}(b)$, that is, both a and b are odd, there must be integers j and k such that $a = 2j + 1$ and $b = 2k + 1$.
> > > In order to show $\text{Odd}(ab)$, we must show there exists some integer l such that $ab = 2l + 1$. But
> > >
> > > $$ab = (2j+1)(2k+1) = 4jk+2j+2k+1 = 2(2jk+j+k)+1,$$
> > >
> > > and therefore $l = 2jk + j + k$ is such an integer.
> >
> > We proved $\text{Odd}(ab)$ under the supposition $\text{Odd}(a) \land \text{Odd}(b)$, so $(\text{Odd}(a) \land \text{Odd}(b)) \to \text{Odd}(ab)$.
>
> Therefore the contrapositive $\text{Even}(ab) \to (\text{Even}(a) \lor \text{Even}(b))$ holds as well.

Since we proved this for arbitrary integers a and b without assuming anything about them, in fact this is true for all integers. □

Exercise 6 Prove: for any positive integer n, n is even if and only if $7n + 4$ is even.

We are asked to prove

$$\forall n : \text{PosInt}.\ \text{Even}(n) \leftrightarrow \text{Even}(7n + 4),$$

which is equivalent to

$$\forall n : \mathbb{Z}.\ (n > 0) \to (\text{Even}(n) \leftrightarrow \text{Even}(7n + 4)).$$

Proof. Let n be an arbitrary integer, and suppose $n > 0$. We will show $(\text{Even}(n) \leftrightarrow \text{Even}(7n + 4))$, by showing the implication in both directions.

> ($\rightarrow$) Suppose $\text{Even}(n)$, that is, suppose there is an integer j such that $n = 2j$. Then we must show $\text{Even}(7n+4)$, that is, $7n+4$ is of the form $2k$ for some integer k.
>
> > $7n + 4 = 7(2j) + 4 = 14j + 4 = 2(7j + 2).$
>
> If $k = (7j+2)$ then $7n+4 = 2k$, so we have shown $7n+4$ is even.

> ($\leftarrow$) In the other direction, we must show $\text{Even}(7n+4) \rightarrow \text{Even}(n)$; we will show the contrapositive, that is, $\text{Odd}(n) \rightarrow \text{Odd}(7n+4)$.
>
> > Suppose $\text{Odd}(n)$, that is, suppose $n = 2j + 1$ for some integer j. We will show $\text{Odd}(7n+4)$.
> >
> > > $7n + 4 = 7(2j+1) + 4 = 14j + 11 = 2(7j+5) + 1$
> >
> > Therefore, $7n+4$ is odd, since it is one more than twice an integer.
>
> We showed $\text{Odd}(n) \rightarrow \text{Odd}(7n+4)$, which is equivalent to its contrapositive, $\text{Even}(7n+4) \rightarrow \text{Even}(n)$.

□

We showed both directions of the biconditional for an arbitrary integer n. Note that we never actually made use of the fact that $n > 0$, so the proof in fact shows that this is true for all integers, not just positive integers.