Discrete Math HW 2: Learning goals L2–L5

L2: I can use algebraic laws to rewrite and simplify, or prove the equivalence of, formulas of propositional logic.

Exercise 1 Each of the following is an **incorrect** attempt to establish a logical equivalence via a series of transformations. For each one, explain the mistake, and give specific values of the propositional variables for which the starting proposition does not have the same truth value as the ending proposition.

(a) $p \lor (p \land q)$

$\equiv$ { Associativity }

$(p \lor p) \land q$

$\equiv$ { Idempotence }

$p \land q$

(b) $p \land (\neg p \lor q)$

$\equiv$ { Distributivity }

$(p \lor \neg p) \land (p \lor q)$

$\equiv$ { Excluded Middle }

$T \land (p \lor q)$

$\equiv$ { Identity }

$p \lor q$

(c) $\neg(q \lor (\neg p \land r))$

$\equiv$ { De Morgan }

$\neg q \lor \neg(\neg p \land r)$

$\equiv$ { Double negation }

$\neg q \lor (p \land r)$

Exercise 2 Use a chain of logical equivalences to show that

$$\neg(\neg(p \land p) \land \neg(q \land q)) \equiv p \lor q.$$

Exercise 3 Use a chain of logical equivalences to show that

$$(p \land q) \to q$$

is a tautology.

Hint: to show that something is a tautology, show that it is logically equivalent to True.

L3: I can use De Morgan laws to simplify negations of propositions with quantifiers, AND, and OR.

Exercise 4 Express the negation of each proposition below so that all negation symbols immediately precede predicate variables. In other words, "push negation inwards as far as possible". For example, given the proposition $\forall x : D.\ P(x) \lor Q(x)$, we could express its negation as

$$\exists x : D. \neg P(x) \land \neg Q(x).$$

Complete at least two to get credit for this exercise.

(a) $\forall x{:}D.\ T(x) \lor G(x) \lor F(x)$

(b) $\forall x{:}D.\ \exists y{:}E.\ P(x) \land Q(x,y) \land R(y)$

(c) $(\exists x{:}\mathbb{Z}.\ P(x)) \lor (\forall y{:}\mathbb{Z}.\ Q(x))$

(d) $\forall x{:}S.\ \exists y{:}D.\ H(x,y) \to (P(x) \lor Q(y))$

(e) $\forall m{:}D.\ \forall n{:}D.\ P(m,n) \leftrightarrow Q(m,n)$

L4: I can formalize English statements as propositional logic formulas, making appropriate use of logical connectives and nested quantifiers.

Exercise 5 For each English sentence below:

- Translate it into formal propositional logic using quantifiers.
- Decide if it is true or false. Justify your answer using a truth table, algebraic reasoning, and/or an informal logical argument.

Complete at least three to get credit.

(a) There is an integer n such that 20 is 7 more than n.

(b) For every integer n, 7 more than n is less than 20.

(c) There are two integers which add to 6.

(d) For any propositions p, q, and r, if p implies q and q implies r, then p implies r.

(e) For every proposition p, there is a proposition q such that $(p \land q) \to \text{True}$.

(f) For every proposition p, there is a proposition q such that $p \land q$ is logically equivalent to True.

Exercise 6 Translate each English sentence below into formal propositional logic, using quantifiers as appropriate. You are welcome to use the predicates $\mathrm{Sq}(n)$ = "n is a perfect square" and $\mathrm{Even}(n)$ = "n is even" which we defined in class.

(a) If n is a perfect square, $n+2$ is not a perfect square.

(b) If mn is even, then either m is even or n is even (or both).

(c) Any integer n is even if and only if $7n+4$ is even.

(d) Every natural number can be written as the sum of two squares of natural numbers.

L5: I can express quantified statements in equivalent ways with different domains.

Exercise 7 Rewrite the below proposition into a logically equivalent proposition using a quantifier with domain $\mathbb{Z}$.

$$\forall x\!:\!\mathbb{N}.\ 7x - 5 > 10$$

Exercise 8 Rewrite the below proposition into a logically equivalent proposition using a quantifier with domain $\mathbb{Z}$.

$$\exists p\!:\!\mathbb{N}.\ p^2 + p = 6$$

Exercise 9 Rewrite the following proposition into a logically equivalent proposition using only quantifiers over $\mathbb{Z}$, that is, using $\forall$ and $\exists$ with domain $\mathbb{Z}$ instead of the more restricted domain Even. You may assume there is a predicate $\mathrm{Even}(n)$ which means "n is even".

$$\forall x\!:\!\mathrm{Even}.\ \exists y\!:\!\mathrm{Even}.\ x = y + 2$$