Discrete Math HW 2: Learning goals L2–L5 (solutions)

L2: I can use algebraic laws to rewrite and simplify, or prove the equivalence of, formulas of propositional logic.

Exercise 1 Each of the following is an **incorrect** attempt to establish a logical equivalence via a series of transformations. For each one, explain the mistake, and give specific values of the propositional variables for which the starting proposition does not have the same truth value as the ending proposition.

(a) $p \lor (p \land q)$
$\equiv$ { Associativity }
$(p \lor p) \land q$
$\equiv$ { Idempotence }
$p \land q$

The Associativity step is incorrect: associativity only applies when we have something like $x \land (y \land z)$ or $x \lor (y \lor z)$. We cannot use it when there is one $\lor$ and one $\land$, as there is here.

Note that the Idempotence step is in fact correct: idempotence tells us that $p \lor p \equiv p$, so it is correct to replace one by the other.

When $p = T$ and $q = F$, we have $p \lor (p \land q) \equiv T$ but $p \land q \equiv F$.

(b) $p \land (\neg p \lor q)$
$\equiv$ { Distributivity }
$(p \lor \neg p) \land (p \lor q)$
$\equiv$ { Excluded Middle }
$T \land (p \lor q)$
$\equiv$ { Identity }
$p \lor q$

The Excluded Middle and Identity steps are correct; the incorrect step is Distributivity, which gets the $\land$ and $\lor$ switched. In fact, distributivity tells us that $p \land (\neg p \lor q) \equiv (p \land \neg p) \lor (p \land q)$.

When $p = F$ and $q = T$, we have $p \land (\neg p \lor q) \equiv F$ but $p \lor q \equiv T$.

(c) $\neg(q \lor (\neg p \land r))$
$\equiv$ { De Morgan }
$\neg q \lor \neg(\neg p \land r)$
$\equiv$ { Double negation }

$\neg q \vee (p \wedge r)$

Both steps are incorrect. In the De Morgan step, the $\vee$ should have switched to $\wedge$. And in the second step, Double Negation does not apply since we do not have $\neg(\neg p)$, but rather $\neg(\neg p \wedge r)$.

When $q = F$ the last proposition is true no matter what p and r are; but when $p = F$ and $r = T$, the first proposition is false.

Exercise 2 Use a chain of logical equivalences to show that

$$\neg(\neg(p \wedge p) \wedge \neg(q \wedge q)) \equiv p \vee q.$$

$\neg(\neg(p \wedge p) \wedge \neg(q \wedge q))$

$\equiv$ { Idempotence (twice) }

$\neg(\neg p \wedge \neg q)$

$\equiv$ { De Morgan }

$\neg(\neg p) \vee \neg(\neg q)$

$\equiv$ { Double negation (twice) }

$p \vee q$

Exercise 3 Use a chain of logical equivalences to show that

$$(p \wedge q) \to q$$

is a tautology.

Hint: to show that something is a tautology, show that it is logically equivalent to True.

$(p \wedge q) \to q$

$\equiv$ { Implication; $a \to b \equiv \neg a \vee b$ }

$\neg(p \wedge q) \vee q$

$\equiv$ { De Morgan }

$(\neg p \vee \neg q) \vee q$

$\equiv$ { Associativity }

$\neg p \vee (\neg q \vee q)$

$\equiv$ { Excluded Middle }

$\neg p \vee T$

$\equiv$ { Annihilation }

T

L3: I can use De Morgan laws to simplify negations of propositions with quantifiers, AND, and OR.

Exercise 4 Express the negation of each proposition below so that all negation symbols immediately precede predicate variables. In other words, "push negation inwards as far as possible". For example, given the proposition $\forall x : D.\ P(x) \lor Q(x)$, we could express its negation as

$$\exists x : D. \neg P(x) \land \neg Q(x).$$

Complete at least two to get credit for this exercise.

(a) $\forall x{:}D.\ T(x) \lor G(x) \lor F(x)$

$$\exists x{:}D.\ \neg T(x) \land \neg G(x) \land \neg F(x)$$

(b) $\forall x{:}D.\ \exists y{:}E.\ P(x) \land Q(x,y) \land R(y)$

$$\exists x{:}D.\ \forall y{:}E.\ \neg P(x) \lor \neg Q(x,y) \lor \neg R(y)$$

(c) $(\exists x{:}\mathbb{Z}.\ P(x)) \lor (\forall y{:}\mathbb{Z}.\ Q(x))$

$$(\forall x{:}\mathbb{Z}.\ \neg P(x)) \land (\exists y{:}\mathbb{Z}.\ \neg Q(x))$$

(d) $\forall x{:}S.\ \exists y{:}D.\ H(x,y) \to (P(x) \lor Q(y))$

Careful! There is no De Morgan law that lets us distribute negation over implication. In fact $\neg(P \to Q) \not\equiv \neg P \to \neg Q$. In order to complete this exercise we must first translate implication via the law $P \to Q \equiv \neg P \lor Q$.

$$\begin{aligned}
& \neg(\forall x{:}S.\ \exists y{:}D.\ H(x,y) \to (P(x) \lor Q(y))) \\
\equiv & \qquad \{ \text{ Implication } \} \\
& \neg(\forall x{:}S.\ \exists y{:}D.\ \neg H(x,y) \lor (P(x) \lor Q(y))) \\
\equiv & \qquad \{ \text{ De Morgan } \} \\
& \exists x{:}S.\ \forall y{:}D.\ \neg\neg H(x,y) \land \neg(P(x) \lor Q(y))) \\
\equiv & \qquad \{ \text{ Double Negation, De Morgan } \} \\
& \exists x{:}S.\ \forall y{:}D.\ H(x,y) \land (\neg P(x) \land \neg Q(y)))
\end{aligned}$$

(e) $\forall m{:}D.\ \forall n{:}D.\ P(m,n) \leftrightarrow Q(m,n)$

We must be careful here too, since $\neg(P \leftrightarrow Q) \not\equiv (\neg P \leftrightarrow \neg Q)$ (in fact, $\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$; showing this is a nice exercise).

$$
\begin{aligned}
& \neg(\forall m:D.\ \forall n:D.\ P(m,n) \leftrightarrow Q(m,n)) \\
\equiv & \qquad \{ \text{ Iff } \} \\
& \neg(\forall m:D.\ \forall n:D.\ (P(m,n) \to Q(m,n)) \land (Q(m,n) \to P(m,n)) \\
\equiv & \qquad \{ \text{ De Morgan, Implication, Double Negation } \} \\
& \exists m:D.\ \exists n:D.\ (P(m,n) \land \neg Q(m,n)) \lor (Q(m,n) \land \neg P(m,n))
\end{aligned}
$$

L4: I can formalize English statements as propositional logic formulas, making appropriate use of logical connectives and nested quantifiers.

Exercise 5 For each English sentence below:

- Translate it into formal propositional logic using quantifiers.
- Decide if it is true or false. Justify your answer using a truth table, algebraic reasoning, and/or an informal logical argument.

Complete at least three to get credit.

(a) There is an integer n such that 20 is 7 more than n.

$$\exists n:\mathbb{Z}.\ 20 = 7 + n$$

This is true: 13 is such a number.

(b) For every integer n, 7 more than n is less than 20.

$$\forall n:\mathbb{Z}.\ 7 + n < 20$$

This is false. As a counterexample, consider $n = 100$: it is not true that $7 + 100 < 20$. Hence $7 + n < 20$ is not true for *all* integers n.

(c) There are two integers which add to 6.

$$\exists x:\mathbb{Z}.\ \exists y:\mathbb{Z}.\ x + y = 6$$

This is true. For example, 2 and 4 are such integers.

(d) For any propositions p, q, and r, if p implies q and q implies r, then p implies r.

$$\forall p:\text{Prop}.\ \forall q:\text{Prop}.\ \forall r:\text{Prop}.\ ((p \to q) \land (q \to r)) \to (p \to r)$$

This is true, *i.e.* implication is transitive. One way to see this is to make a truth table.

(e) For every proposition p, there is a proposition q such that $(p \wedge q) \to \text{True}$.

$$\forall p : \text{Prop}.\ \exists q : \text{Prop}.\ (p \wedge q) \to T$$

This is true: we can always pick $q = F$. Then no matter what p is, $(p \wedge q) \to T \equiv F \to T \equiv T$.

(f) For every proposition p, there is a proposition q such that $p \wedge q$ is logically equivalent to True.

$$\forall p : \text{Prop}.\ \exists q : \text{Prop}.\ (p \wedge q) \equiv T$$

This is false. As a counterexample, consider $p = F$: there is no way to pick a proposition q such that $F \wedge q \equiv T$.

Exercise 6 Translate each English sentence below into formal propositional logic, using quantifiers as appropriate. You are welcome to use the predicates $\text{Sq}(n)$ = "n is a perfect square" and $\text{Even}(n)$ = "n is even" which we defined in class.

(a) If n is a perfect square, $n + 2$ is not a perfect square.

$$\forall n : \mathbb{N}.\ \text{Sq}(n) \to \neg \text{Sq}(n + 2)$$

Remember that when a statement talks about some variable without defining it, we should usually interpret it as a $\forall$.

(b) If mn is even, then either m is even or n is even (or both).

$$\forall m : \mathbb{Z}.\ \forall n : \mathbb{Z}.\ \text{Even}(mn) \to (\text{Even}(m) \vee \text{Even}(n))$$

(c) Any integer n is even if and only if $7n + 4$ is even.

$$\forall n : \mathbb{Z}.\ \text{Even}(n) \leftrightarrow \text{Even}(7n + 4)$$

(d) Every natural number can be written as the sum of two squares of natural numbers.

$$\forall n : \mathbb{N}.\ \exists p : \mathbb{N}.\ \exists q : \mathbb{N}.\ \text{Sq}(p) \wedge \text{Sq}(q) \wedge (n = p + q)$$

OR

$$\forall n : \mathbb{N}.\ \exists p : \mathbb{N}.\ \exists q : \mathbb{N}.\ (n = p^2 + q^2)$$

Note that it does not make sense to write something like $\text{Sq}(p) + \text{Sq}(q)$, since this is a type error: $\text{Sq}(p)$ is a proposition, i.e. a true-or-false thing, not a number that can be added.

L5: I can express quantified statements in equivalent ways with different domains.

Exercise 7 Rewrite the below proposition into a logically equivalent proposition using a quantifier with domain $\mathbb{Z}$.

$$\forall x : \mathbb{N}.\ 7x - 5 > 10 \equiv \forall x : \mathbb{Z}.\ (n \geq 0) \to (7x - 5 > 10)$$

Exercise 8 Rewrite the below proposition into a logically equivalent proposition using a quantifier with domain $\mathbb{Z}$.

$$\exists p : \mathbb{N}.\ p^2 + p = 6 \equiv \exists p : \mathbb{Z}.\ (p \geq 0) \wedge (p^2 + p = 6)$$

Exercise 9 Rewrite the following proposition into a logically equivalent proposition using only quantifiers over $\mathbb{Z}$, that is, using $\forall$ and $\exists$ with domain $\mathbb{Z}$ instead of the more restricted domain Even. You may assume there is a predicate $\text{Even}(n)$ which means "n is even".

$$\forall x : \text{Even}.\ \exists y : \text{Even}.\ x = y + 2 \equiv \forall x : \mathbb{Z}.\ \text{Even}(x) \to (\exists y : \mathbb{Z}.\ \text{Even}(y) \wedge x = y + 2)$$